Vector Spaces

Jan Schneider

McCombs School of Business
University of Texas at Austin

Vectors

$\mathbb{R}$: set of real numbers (scalars).

$\mathbb{R}^n$: set of all vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad x_i \in \mathbb{R}.$$

$X \subseteq \mathbb{R}^n$ is a vector space or linear space if for any $\mathbf{x}, \mathbf{y} \in X$ and $c \in \mathbb{R}$:

$$\mathbf{x} + \mathbf{y} \in X \qquad \text{and } c\mathbf{x} \in X.$$

Y is a subspace of X ($Y \subseteq X$) if Y is a vector space and if every element of Y is also in X.

Inner Product

Let $X \subseteq \mathbb{R}^n$ be a vector space.

An inner product is a function $f(\mathbf{x}, \mathbf{y}) : X^2 \to \mathbb{R}$, such that for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ and $a, b \in \mathbb{R}$:

1. $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$
2. $f(\mathbf{x}, (a\mathbf{y} + b\mathbf{z})) = af(\mathbf{x}, \mathbf{y}) + bf(\mathbf{x}, \mathbf{z})$
3. $\mathbf{x} \neq 0 \Rightarrow f(\mathbf{x}, \mathbf{x}) > 0$

Example:

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

is called the Euclidian product.

Linear Combinations

If $c_i, \ldots, c_k \in \mathbb{R}$, and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ then

$$\mathbf{Ac} = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = c_1 \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} + \cdots + c_k \begin{pmatrix} x_{1k} \\ \vdots \\ x_{nk} \end{pmatrix} = \sum_{i=1}^{k} c_i \mathbf{x}_i$$

is a linear combination.

Matrix Multiplication I

Alternatively, we can also write a linear combination as

$$\mathbf{Ab} = \begin{pmatrix} \mathbf{by}_1 \\ \vdots \\ \mathbf{by}_n \end{pmatrix}, \qquad \text{where } \mathbf{y}_1 \ldots \mathbf{y}_n \text{ are the row vectors of } \mathbf{A}.$$

And accordingly for matrix multiplication:

$$\mathbf{AB} = \begin{pmatrix} \mathbf{b}_1\mathbf{y}_1 & \mathbf{b}_2\mathbf{y}_1 & \cdots & \mathbf{b}_m\mathbf{y}_1 \\ \vdots & \vdots & & \vdots \\ \mathbf{b}_1\mathbf{y}_n & \mathbf{b}_2\mathbf{y}_n & \cdots & \mathbf{b}_m\mathbf{y}_n \end{pmatrix}$$

Matrix Multiplication II

Let $f(\cdot,\cdot)$ be a weighted Euclidian product:

$$f(\mathbf{A},\mathbf{B}) = \begin{pmatrix} f(\mathbf{b}_1,\mathbf{y}_1) & f(\mathbf{b}_2,\mathbf{y}_1) & \cdots & f(\mathbf{b}_m,\mathbf{y}_1) \\ \vdots & \vdots & & \vdots \\ f(\mathbf{b}_1,\mathbf{y}_n) & f(\mathbf{b}_2,\mathbf{y}_n) & \cdots & f(\mathbf{b}_m,\mathbf{y}_n) \end{pmatrix}, \qquad f(\mathbf{b}_i,\mathbf{y}_j) = \sum_{s=1}^{k} c_s \mathbf{b}_{is}\, \underbrace{\mathbf{y}_{js}}_{\text{element } s \text{ of vector } \mathbf{y}_j}$$

Hence the weighted multiplication of matrixes **A** and **B** is identical to regular multiplication of an adjusted matrix $\tilde{\mathbf{A}}$ with **B**:

$$f(\mathbf{A},\mathbf{B}) = \underbrace{\begin{pmatrix} c_1 a_{11} & \cdots & c_k a_{1k} \\ \vdots & & \vdots \\ c_1 a_{n1} & \cdots & c_k a_{nk} \end{pmatrix}}_{\tilde{\mathbf{A}}} \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{k1} & \cdots & b_{km} \end{pmatrix} = \tilde{\mathbf{A}} \cdot \mathbf{B}$$

A set of equations

$$\mathbf{Ab} = \mathbf{c} \qquad \iff \qquad \sum_{i=1}^{k} b_i \mathbf{x}_i = \mathbf{c}$$

is a linear system.

Vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent if

$$\mathbf{Ab} = \sum_i b_i \mathbf{x}_i = \mathbf{0} \qquad \Longrightarrow \qquad b_1 = \cdots = b_k = 0.$$

Example: does $\mathbf{A}^T\mathbf{A}$ consist of independent vectors? We have:

$$\begin{aligned} & \mathbf{0} = \mathbf{A}^T\mathbf{A}\mathbf{c} \\ \Longrightarrow \qquad & \mathbf{c}^T\mathbf{0} = \mathbf{c}^T\mathbf{A}^T\mathbf{A}\mathbf{c} = (\mathbf{A}\mathbf{c})^T(\mathbf{A}\mathbf{c}) \\ \Longrightarrow \qquad & \mathbf{A}\mathbf{c} = \mathbf{0} \end{aligned}$$

Basis

- number of independent vectors in matrix $\mathbf{A}$: rank of $\mathbf{A}$
- all vectors in A are independent: $\mathbf{A}$ has full rank
- if X is the set of all linear combinations of $Y \subset \mathbb{R}^n$, then X is the span of Y (Y spans X)
- An independent subset of X which spans X is a basis of X
- For $\mathbb{R}^n$, the set of vectors contained in the identity matrix with dimension n is called the standard basis.

Orthogonal Vectors

Consider a vector space X and an inner product $f(\mathbf{x}, \mathbf{y})$.

We say vectors $\mathbf{x}, \mathbf{y} \in X$ are orthogonal ($\mathbf{x} \perp \mathbf{y}$) if $f(\mathbf{x}, \mathbf{y}) = 0$.

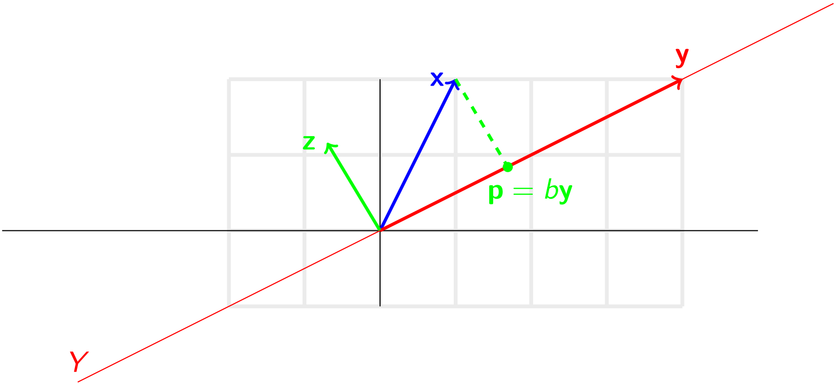
Orthogonal vectors are independent:

$$\left.\begin{aligned} a\mathbf{x} + b\mathbf{y} &= 0 \\ \mathbf{x}\mathbf{y} &= 0 \end{aligned}\right\} \implies \begin{aligned} a\,\overbrace{\mathbf{x}\mathbf{x}}^{>0 \text{ if } \mathbf{x} \neq 0} + b\,\overbrace{\mathbf{x}\mathbf{y}}^{=0} &= 0 \\ a\,\underbrace{\mathbf{y}\mathbf{x}}_{=0} + b\,\underbrace{\mathbf{y}\mathbf{y}}_{>0 \text{ if } \mathbf{y} \neq 0} &= 0 \end{aligned} \implies a = b = 0$$

Orthogonal complement to Y:

$$Y_{\perp} = \{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) = 0 \text{ for all } \mathbf{y} \in Y\}$$

Orthogonal Projection: 2 Dimensions



Orthogonal Projection: 2 Dimensions

Define

$$\mathbf{z} = \underbrace{\mathbf{x} - b\mathbf{y}}_{\text{green vector}}$$

Since $\mathbf{z}$ and $\mathbf{y}$ are orthogonal:

$$\mathbf{z}\mathbf{y} = (\mathbf{x} - b\mathbf{y})\mathbf{y} = 0 \qquad \Longleftrightarrow \qquad b = \frac{\mathbf{x}\mathbf{y}}{\mathbf{y}\mathbf{y}}$$

and

$$\underbrace{\mathbf{x}}_{\in X} = \underbrace{b\mathbf{y}}_{\in Y} + \underbrace{\mathbf{z}}_{\in Y_\perp}$$

We can generalize the concept of orthogonal projection to an arbitrary inner product:

$$f(\mathbf{x} - b\mathbf{y}, \mathbf{y}) = 0 \qquad \Longleftrightarrow \qquad b = \frac{f(\mathbf{x}, \mathbf{y})}{f(\mathbf{y}, \mathbf{y})}$$

Orthogonal Projection: n Dimensions

Let $X \subseteq \mathbb{R}^n$ be a vector space and Y be a subspace of X.

Let $\mathbf{A}$ be an $n \times m$ matrix whose m column vectors form a basis for Y.

For any $\mathbf{x} \in X$ define

$$\underbrace{\mathbf{b}}_{m \times 1} = \underbrace{(\overbrace{\mathbf{A}^T\mathbf{A}}^{m \times m})^{-1} \overbrace{\mathbf{A}^T}^{m \times n}}_{m \times n} \underbrace{\mathbf{x}}_{n \times 1}, \qquad \underbrace{\mathbf{z}}_{n \times 1} = \mathbf{x} - \mathbf{A}\mathbf{b}$$

Orthogonal Projection: n Dimensions

Then

$$\mathbf{x} = \mathbf{A}\mathbf{b} + \mathbf{z}$$

and

$$\mathbf{A}^T\mathbf{x} = \underbrace{\mathbf{A}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{x}}_{\mathbf{A}^T\mathbf{x}} + \mathbf{A}^T\mathbf{z} \qquad \Longrightarrow \qquad \mathbf{A}^T\mathbf{z} = 0$$

Hence

$$\underbrace{\mathbf{x}}_{\in X} = \underbrace{\mathbf{A}\mathbf{b}}_{\in Y} + \underbrace{\mathbf{z}}_{\in Y_\perp}$$

Orthogonal Projection: Weighted Euclidian Product I

Define:

$$\mathbf{b} = f(\mathbf{A}^T, \mathbf{A})^{-1} \cdot f(\mathbf{A}^T, \mathbf{x}), \qquad \mathbf{z} = \mathbf{x} - \mathbf{A} \cdot \mathbf{b}$$

regular product

where:

$$f(\mathbf{A}, \mathbf{B}) = \underbrace{\tilde{\mathbf{A}}}_{\text{matrix } \mathbf{A} \text{ adjusted by weights } c_i} \cdot \mathbf{B}$$

Hence:

$$f\left(\mathbf{A}^T, \mathbf{A} \cdot f(\mathbf{A}^T, \mathbf{A})^{-1} \cdot f(\mathbf{A}^T, \mathbf{x})\right) = \tilde{\mathbf{A}^T} \cdot \left(\mathbf{A} \cdot f(\mathbf{A}^T, \mathbf{A})^{-1} \cdot f(\mathbf{A}^T, \mathbf{x})\right)$$

$$= \underbrace{\underbrace{(\tilde{\mathbf{A}^T} \cdot \mathbf{A})}_{f(\mathbf{A}^T, \mathbf{A})} \cdot f(\mathbf{A}^T, \mathbf{A})^{-1} \cdot f(\mathbf{A}^T, \mathbf{x})}_{f(\mathbf{A}^T, \mathbf{x})}$$

Orthogonal Projection: Weighted Euclidian Product II

Hence

$$f(\mathbf{A}^T, \mathbf{x}) = \underbrace{f\left(\mathbf{A}^T, \mathbf{A} \cdot f(\mathbf{A}^T, \mathbf{A})^{-1} \cdot f(\mathbf{A}^T, \mathbf{x})\right)}_{f(\mathbf{A}^T, \mathbf{x})} + f(\mathbf{A}^T, \mathbf{z}) \implies f(\mathbf{A}^T, \mathbf{z}) = 0$$

Hence we can decompose vector $\mathbf{x}$ as

$$\underbrace{\mathbf{x}}_{\in X} = \underbrace{\mathbf{Ab}}_{\in Y} + \underbrace{\mathbf{z}}_{\in Y_\perp} \text{ with respected to the weighted product } f(\mathbf{x}, \mathbf{y})$$