Structure of Uncertainty

Jan Schneider

McCombs School of Business
University of Texas at Austin

States of the World

- countably infinite number of states of the world
- example: rational numbers in $[0, 1]$
- set of states:

$$S = \{s_1, s_2, s_3, \ldots\}$$

- nature chooses state $s \in S$ at beginning of time
- individuals learn slowly more about true state

Partitions

- partition $Q = \{q_1, \cdots, q_N\}$ of $S = \{s_1, s_2, \cdots\}$ is a collection of subsets of S such that

$$q_i \cap q_j = \emptyset \text{ for } j \neq i \quad \text{and} \quad \cup_{j=1}^{N} q_j = \mathbf{S}$$

- assumption: partitions contain a finite number of elements
- example:

$$S = \{ \underbrace{\text{rationals} \in [0, 0.2),}_{q_1} \underbrace{\text{rationals} \in [0.2, 0.6),}_{q_2} \underbrace{\text{rationals} \in [0.6, 1]}_{q_3} \}$$

Potential Information = Partition

- example (partition from previous slide):

if nature chooses $s =$	then at time t we know that $s \in$
0.05	$[0, 0.2)$
0.1	$[0, 0.2)$
0.5	$[0.2, 0.6)$
0.8	$[0.6, 1]$
1	$[0.6, 1]$

Information and Time

- example: time $t+1$ information

$$S = \{ \underbrace{[0, 0.2),}_{q_1} \underbrace{[0.2, 0.6],}_{q_2} \underbrace{(0.6, 0.9),}_{q_3} \underbrace{[0.9, 1]}_{q_4} \}$$

- then information evolves as follows:

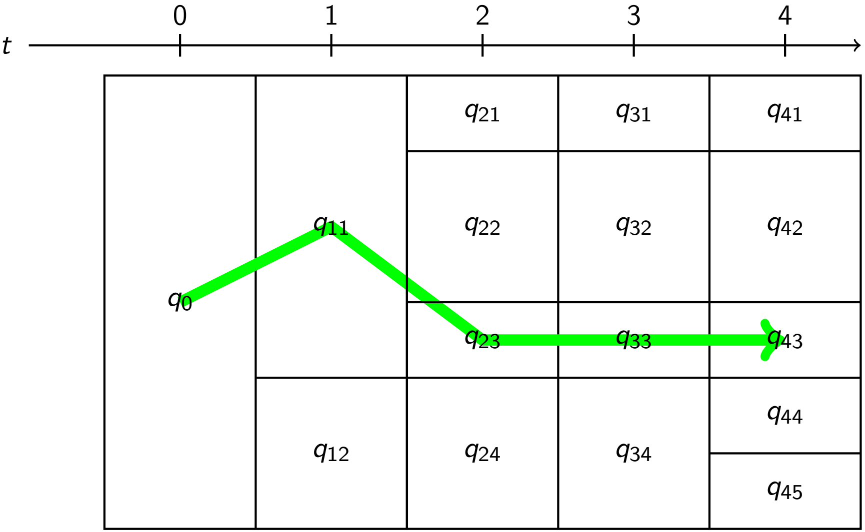
if nature chooses $s =$	time t info: $s \in$	time $t+1$ info: $s \in$	
$1/20$	$[0, 0.2)$	$[0, 0.2)$	
$1/10$	$[0, 0.2)$	$[0, 0.2)]$	
$1/2$	$[0.2, 0.6]$	$[0.2, 0.6]$	
$3/4$	$(0.6, 1]$	$(0.6, 0.9)$	$\leftarrow$ new info
1	$(0.6, 1]$	$[0.9, 1]$	$\leftarrow$ new info

Filtration

- information structure at time t: $Q_t = \{q_{t1}, \ldots, q_{tN_t}\}$
- no forgetting $\Rightarrow$ partition becomes finer over time
- sequence of finer partitions $\{Q_0, Q_1, Q_2 \ldots\}$: filtration
- each state $s \in S$ defines a unique path through the partition elements of this filtration:

$$S \supseteq q_0 \supseteq q_1 \supseteq q_2 \cdots$$

Example of a Filtration



Random Variables and Information I

- random variable: function that assigns a real number to each state $s \in S$
- Random variable Y is measurable with respect to partition Q if

$$s_i \in q \text{ and } s_j \in q \quad \Longrightarrow \quad Y(s_i) = Y(s_j) \qquad \text{for all } q \in Q. \tag{1}$$

- stochastic process: sequence of random variables

$$\{Y_0, Y_1, Y_2, \dots\}$$

- stochastic process $\{Y_0, Y_1, Y_2, \dots\}$ is adapted to a filtration $\{Q_0, Q_1, Q_2 \dots\}$ if each Y_t is measurable with respect to Q_t

Adapted Process: Example

state s	Q_t	X_t	Y_t	Z_t	Q_{t+1}	X_{t+1}	Y_{t+1}	Z_{t+1}
$1/10$	$[0, 0.2)$	4	3	1	$[0, 0.2)$	3	7	1
$1/20$	$[0, 0.2)$	4	3	2	$[0, 0.2)$	3	7	1
$1/2$	$[0.2, 0.6)$	4	4	1	$[0.2, 0.6)$	4	8	2
$3/4$	$[0.6, 1]$	2	5	1	$[0.6, 0.9]$	4	9	2
1	$[0.6, 1]$	2	5	1	$(0.9, 1]$	1	10	2

Random Variables and Information II

- stochastic process $\{Y_t, Y_{t+1}\}$ generates the filtration $\{Q_t, Q_{t+1}\}$ if:

$$s_i, s_j \in q_t \quad \iff \quad Y_t(s_i) = Y_t(s_j)$$

Probability Measure

A probability measure “prob” is a function from the power set of S to the real numbers with the following properties:

1. $A \subseteq S \implies \text{prob}[A] \geq 0$
2. $\text{prob}[S] = 1$
3. $A_i \cap A_j = \emptyset$ for $j \neq i \quad \implies \quad \text{prob}\left[\cup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \text{prob}[A_n]$

Conditional Probabilities

- probability that nature chooses a state s in q_t: $\text{prob}(q_t)$
- assume: $\text{prob}(q_t) > 0$
- probability of $q_{t+\tau} \subseteq q_t$ conditional on observing q_t:

$$\text{prob}(q_{t+\tau}|q_t) = \frac{\text{prob}(q_{t+\tau})}{\text{prob}(q_t)}$$

- we can also write this probability as

$$\text{prob}(q_{t+\tau}|q_t) = \underbrace{\frac{\text{prob}(q_{t+1})}{\text{prob}(q_t)}}_{\text{prob}(q_{t+1}|q_t)} \times \underbrace{\frac{\text{prob}(q_{t+2})}{\text{prob}(q_{t+1})}}_{\text{prob}(q_{t+2}|q_{t+1})} \times \cdots \times \underbrace{\frac{\text{prob}(q_{t+\tau})}{\text{prob}(q_{t+\tau-1})}}_{\text{prob}(q_{t+\tau}|q_{t+\tau-1})}$$

Expectation

- unconditional expectation of a random variables Y_t:

$$\mathrm{E}[Y_t] = \sum_{q_t \subseteq Q_t} \mathrm{prob}(q_t) \times Y_t(q_t)$$

- conditional:

$$\mathrm{E}[Y_{t+\tau}|q_t] = \sum_{q_{t+\tau} \subseteq q_t} \mathrm{prob}(q_{t+\tau}|q_t) \times Y(q_{t+\tau})$$

- $E[Y_{t+\tau}|Q_t]$: random variable with realizations

$$\left\{ \mathrm{E}[Y_{t+\tau}|q_{1t}],\ \mathrm{E}[Y_{t+\tau}|q_{2t}],\ \ldots\ ,\ \mathrm{E}[Y_{t+\tau}|q_{N_t t}] \right\}$$

- usually abbreviate:

$$E_t[Y_{t+\tau}] = E[Y_{t+\tau}|Q_t]$$

Iterated Expectation I

- consider the following sequence of expectations:

t	$\cdots$	$t+\tau_1$	$\cdots$	$t+\tau_2$
$\mathrm{E}\Big[\mathrm{E}[Y_{t+\tau_2} \mid Q_{t+\tau_1}]\Big\vert q_t\Big]$		$\mathrm{E}[Y_{t+\tau_2} \mid Q_{t+\tau_1}]$		$Y_{t+\tau_2}$

Iterated Expectation II

$$\sum_{q_{t+\tau_2} \subset q_{t+\tau_1}} \text{prob}(q_{t+\tau_2} | q_{t+\tau_1}) \, Y(q_{t+\tau_2})$$

$$\mathrm{E}\Big[\mathrm{E}[Y_{t+\tau_2} | Q_{t+\tau_1}] \Big| q_t\Big] = \sum_{q_{t+\tau_1} \subseteq q_t} \text{prob}(q_{t+\tau_1} | q_t) \, \overbrace{\mathrm{E}[Y_{t+\tau_2} | q_{t+\tau_1}]}$$

$$= \underbrace{\sum_{q_{t+\tau_1} \subseteq q_t} \sum_{q_{t+\tau_2} \subset q_{t+\tau_1}} \underbrace{\text{prob}(q_{t+\tau_1} | q_t)}_{\frac{\text{prob}(q_{t+\tau_1})}{\text{prob}(q_t)}} \underbrace{\text{prob}(q_{t+\tau_2} | q_{t+\tau_1})}_{\frac{\text{prob}(q_{t+\tau_2})}{\text{prob}(q_{t+\tau_1})}}}_{\sum_{q_{t+\tau_2} \subseteq q_t} \text{prob}(q_{t+\tau_2} | q_t)} \, Y(q_{t+\tau_2})$$

$$= \mathrm{E}[Y_{t+\tau_2} | q_t]$$