State Prices

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Definition of State Prices

- intuition for state prices: how much does it cost to receive a payoff of 1 at some future q_t?
- For each local market q_t choose a vector of real numbers

$$\boldsymbol{\pi}_{q_{t+1}}(q_t) = \begin{pmatrix} \pi_{q_{1t+1}}(q_t) \\ \vdots \\ \pi_{q_{mt+1}}(q_t) \end{pmatrix}$$

- We call the elements of $\boldsymbol{\pi}_{q_{t+1}}(q_t)$ state prices if

$$\underbrace{\mathbf{P}_t(q_t)}_{\text{asset prices}} = \underbrace{\left(\mathbf{P}^*_{t+1}(q_{t+1})\right)^T}_{\text{payoffs}} \underbrace{\boldsymbol{\pi}_{q_{t+1}}(q_t)}_{\text{state prices}}$$

Longterm State Prices

- we define the long-term state prices as

$$\pi_{q_{t+\tau}}(q_t) = \pi_{q_{t+1}}(q_t) \times \pi_{q_{t+2}}(q_{t+1}) \times \cdots \times \pi_{q_{t+\tau}}(q_{t+\tau-1})$$

$$\sum_{q_{t+2}\subseteq q_{t+1}} \pi_{q_{t+2}}(q_{t+1}) \times [P_a(q_{t+2}) + D_a(q_{t+2})]$$

$$P_a(q_t) = \sum_{q_{t+1}\subset q_t} \pi_{q_{t+1}}(q_t) \times [P_a(q_{t+1}) + D_a(q_{t+1})]$$

$$= \sum_{q_{t+1}\subseteq q_t} \pi_{q_{t+1}}(q_t) D_a(q_{t+1})$$

$$+ \sum_{q_{t+1}\subseteq q_t} \pi_{q_{t+1}}(q_t) \sum_{q_{t+2}\subseteq q_{t+1}} \pi_{q_{t+2}}(q_{t+1}) \times [P_a(q_{t+2}) + D_a(q_{t+2})]$$

$$\sum_{q_{t+1}\subset q_t} \sum_{q_{t+2}\subseteq q_{t+1}} \pi_{q_{t+1}}(q_t)\pi_{q_{t+2}}(q_{t+1})$$

$$\pi_{q_{t+2}}(q_t)$$

$$\sum_{q_{t+2}\subset q_t} \pi_{q_{t+2}}(q_t)$$

- if we continue to solve this equation forward:

$$P_a(q_t) = \sum_{j=1}^{\tau-1} \sum_{q_{t+j} \subseteq q_t} \pi_{q_{t+j}}(q_t) D_a(q_{t+j}) + \sum_{q_{t+\tau} \subseteq q_t} \pi_{q_{t+\tau}}(q_t)[P_a(q_{t+\tau}) + D_a(q_{t+\tau})]$$

State Prices and Portfolios

- we have
$$\sum_{q_{t+1}\subseteq q_t} \pi_{q_{t+1}}(q_t) \times [P_a(q_{t+1}) + D_a(q_{t+1})]$$

$$P_H(q_t) = \sum_a H_a(q_t) \overbrace{P_a(q_t)}$$

$$= \sum_{q_{t+1}\subseteq q_t} \pi_{q_{t+1}}(q_t) \underbrace{\left(\sum_a H_a(q_t) \times [P_a(q_{t+1}) + D_a(q_{t+1})] \right)}_{D_H(q_{t+1}) + P_H(q_{t+1})}$$

- hence:

$$P_H(q_t) = \sum_{j=1}^{\tau-1} \sum_{q_{t+j}\subseteq q_t} \pi_{q_{t+j}}(q_t) D_H(q_{t+j}) + \sum_{q_{t+\tau}\subset q_t} \pi_{q_{t+\tau}}(q_t)[P_H(q_{t+\tau}) + D_H(q_{t+\tau})]$$

State Prices and Risk Free Rates

$$P_f^{t+\tau}(q_t) = \sum_{q_{t+\tau} \subseteq q_t} \pi_{q_{t+\tau}}(q_t) \qquad \Longleftrightarrow \qquad R_f^{t+\tau}(q_t) = \frac{1}{\sum_{q_{t+\tau} \subseteq q_t} \pi_{q_{t+\tau}}(q_t)}$$

Law of One Price $\Longrightarrow$ State Prices Exist

- complete market

$$\mathbf{P}_t = (\mathbf{P}^*_{t+1})^T \underbrace{((\mathbf{P}^*_{t+1})^T)^{-1}\mathbf{P}_t}_{\mathbf{P}_{qt+1}(q_t) = \text{prices of state assets}}$$

- in general, law of one price:

$$\mathbf{P}_t = (\mathbf{P}^*_{t+1})^T \mathbf{P}^*_{t+1}\left((\mathbf{P}^*_{t+1})^T \mathbf{P}^*_{t+1}\right)^{-1}\mathbf{P}_t$$

- Hence

$$\boldsymbol{\pi} = \mathbf{P}^*_{t+1}\left((\mathbf{P}^*_{t+1})^T \mathbf{P}^*_{t+1}\right)^{-1}\mathbf{P}_t$$

  is a state-price vector.

Interpretation of State-Price Vector I

- orthogonal projection:

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{P}^*_{t+1} \underbrace{\mathbf{H}}_{\left((\mathbf{P}^*_{t+1})^T \mathbf{P}^*_{t+1}\right)^{-1} (\mathbf{P}^*_{t+1})^T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} + \overbrace{\mathbf{z}}^{\in \text{ (payoff space)}_\perp}$$

Interpretation of State-Price Vector II

- Accordingly:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \overbrace{\underbrace{\mathbf{P}^*_{t+1}\left((\mathbf{P}^*_{t+1})^T\mathbf{P}^*_{t+1}\right)^{-1}(\mathbf{P}^*_{t+1})^T \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}}_{(\mathbf{H}_1 \ldots \mathbf{H}_m)}}^{\text{traded payoffs closest to state assets}} + (\mathbf{z}_1 \ldots \mathbf{z}_m)$$

Interpretation of State-Price Vector III

- Prices of portfolios whose payoffs are closest to state assets:

$$\underbrace{\left(\mathbf{H}_1 \ldots \mathbf{H}_m\right)^T}_{m \times A} \underbrace{\mathbf{P}_t}_{A \times 1} = \mathbf{P}^*_{t+1}\left(\left(\mathbf{P}^*_{t+1}\right)^T \mathbf{P}^*_{t+1}\right)^{-1} \mathbf{P}_t$$

Complete Market $\iff$ Unique State Prices

- Let $\boldsymbol{\pi}$ be a state-price vector
- Suppose $\mathbf{z}_{P^*_\perp} \in$ (payoff space)$_\perp$
- then:

$$\left(\mathbf{P}^*_{t+1}\right)^T\left(\boldsymbol{\pi} + \mathbf{z}_{P^*_\perp}\right) = \left(\mathbf{P}^*_{t+1}\right)^T\boldsymbol{\pi} + \underbrace{\left(\mathbf{P}^*_{t+1}\right)^T\mathbf{z}_{P^*_\perp}}_{=0} = \mathbf{P}_t$$

- Hence:

$$\boldsymbol{\pi} \text{ is a stare-price vector} \iff (\boldsymbol{\pi} + \underbrace{\mathbf{z}}_{\in \text{(payoff space)}_\perp}) \text{ is a state-price vector}$$

- Hence:

$$\text{market complete} \iff \text{state prices are unique}$$

There is Only One Traded State Price Vector

- Suppose there are two traded state price vectors $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$:

$$
\begin{aligned}
&\text{price of } \mathbf{x} = \mathbf{x}\boldsymbol{\pi}_1 = \mathbf{x}\boldsymbol{\pi}_2 \\
&\Longrightarrow \quad 0 = \mathbf{x}(\boldsymbol{\pi}_1 - \boldsymbol{\pi}_2) \quad \longleftarrow \text{ for any traded payoff } \mathbf{x} \\
&\Longrightarrow \quad 0 = (\boldsymbol{\pi}_1 - \boldsymbol{\pi}_2)(\boldsymbol{\pi}_1 - \boldsymbol{\pi}_2) \quad \longleftarrow \text{ since } \boldsymbol{\pi}_1 - \boldsymbol{\pi}_2 \text{ is also traded} \\
&\Longrightarrow \quad 0 = \boldsymbol{\pi}_1 - \boldsymbol{\pi}_2
\end{aligned}
$$

- hence we can write every state price vector as:

$$
\boldsymbol{\pi} = \underbrace{\mathbf{P}^*_{t+1}\left((\mathbf{P}^*_{t+1})^T \mathbf{P}^*_{t+1}\right)^{-1}\mathbf{P}_t}_{\text{unique traded state price vector}} + \underbrace{\mathbf{z}}_{\in (\text{payoff space})_\perp}
$$

Non-Traded Payoffs Do Not Have a Unique Price

- We can decompose payoff $\mathbf{x}$:

$$\mathbf{x} = \mathbf{x}_{\mathbf{P}^*} + \mathbf{x}_{\mathbf{P}^*_\perp}$$

- We can decompose payoff state-price vector $\boldsymbol{\pi}$:

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{\mathbf{P}^*} + \boldsymbol{\pi}_{\mathbf{P}^*_\perp}$$

- Hence:

$$\text{price of } \mathbf{x} = \mathbf{x}(\boldsymbol{\pi}_{\mathbf{P}^*} + \boldsymbol{\pi}_{\mathbf{P}^*_\perp}) = \mathbf{x}_{\mathbf{P}^*}\boldsymbol{\pi}_{\mathbf{P}^*} + \mathbf{x}_{\mathbf{P}^*_\perp}\boldsymbol{\pi}_{\mathbf{P}^*_\perp} + \underbrace{\mathbf{x}_{\mathbf{P}^*}\boldsymbol{\pi}_{\mathbf{P}^*_\perp} + \boldsymbol{\pi}_{\mathbf{P}^*}\mathbf{x}_{\mathbf{P}^*_\perp}}_{=0}$$

- Suppose we choose $\boldsymbol{\pi}_{\mathbf{P}^*_\perp} = k\mathbf{x}_{\mathbf{P}^*_\perp}$. Then

$$\text{price of } \mathbf{x} = \mathbf{x}_{\mathbf{P}^*}\boldsymbol{\pi}_{\mathbf{P}^*} + k\mathbf{x}^2_{\mathbf{P}^*_\perp}.$$

- Hence

$$\mathbf{x} \in \text{ payoff space} \quad \Longleftrightarrow \quad \text{price of } \mathbf{x} \text{ is constant across all state prices}$$