SDF - General Implications for Prices and Returns

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Traded Discount Factor

- Project discount factor on the payoff space:

$$M_{t+1} = M_{\mathrm{p}t+1} + M_{\mathrm{p}_\perp t+1}$$

$$\Longrightarrow \qquad P_{at} = E_t[M_{t+1}(P_{at+1} + D_{at+1})] = E_t[M_{\mathrm{p}t+1}(P_{at+1} + D_{at+1})]$$

$$\Longleftrightarrow \qquad 1 = \mathrm{E}_t[M_{\mathrm{p}t+1} R_{t+1}]$$

Traded Discount Factor is Unique

- suppose there are two traded discount factors:

$$
\begin{aligned}
E_t[M_{\mathrm{p}t+1}(P_{at+1} + D_{at+1})] &= E_t[\tilde{M}_{\mathrm{p}t+1}(P_{at+1} + D_{at+1})] \\
\Longrightarrow \qquad 0 &= E_t[(M_{\mathrm{p}t+1} - \tilde{M}_{\mathrm{p}t+1})(P_{at+1} + D_{at+1})] \\
\Longrightarrow \qquad 0 &= E_t[(M_{\mathrm{p}t+1} - \tilde{M}_{\mathrm{p}t+1})(M_{\mathrm{p}t+1} - \tilde{M}_{\mathrm{p}t+1})] \\
\Longrightarrow \qquad 0 &= M_{\mathrm{p}t+1} - \tilde{M}_{\mathrm{p}t+1}
\end{aligned}
$$

Longterm Traded Discount Factor

- longterm compound return:

$$1 = \mathrm{E}_t[(M_{\mathrm{p}t+1} \cdots M_{\mathrm{p}t+\tau-1}) R_t^{t+\tau}]$$

- longterm payoff space:

$$(\text{longterm payoff space})_{q_t}^{q_{t+\tau}}$$
$$= \{\mathbf{x} : \mathbf{x} = \mathbf{P}_H(q_{t+\tau}) + \mathbf{D}_H(q_{t+\tau}) \text{ for some self-financing portfolio strategy } (\mathbf{H}_{q_t} \ldots \mathbf{H}_{q_{t+\tau-1}})\}$$

- projecting $M_t^{t+\tau}$ on the corresponding longterm payoff space:

$$M_t^{t+\tau} = \underbrace{M_{\mathrm{p}t}^{t+\tau}}_{t+\tau \text{ payoff of a portfolio that is self-financing between } t \text{ and } t+\tau} + M_{\mathrm{p}_\perp t}^{t+\tau} \quad \Longrightarrow \quad 1 = \mathrm{E}_t[M_t^{t+\tau} R_{at}^{t+\tau}] = \mathrm{E}_t[M_{\mathrm{p}t}^{t+\tau} R_{at}^{t+\tau}]$$

Log-Normal Distribution

- Suppose $Y \sim N(\mu, \sigma^2)$. Let $Z = e^Y$. Then $\log Z \sim N(\mu, \sigma^2)$. We say Z is log-normally distributed. We have

$$\begin{aligned} & \mathrm{E}[Z] = e^{\mu + \frac{\sigma^2}{2}} \\ \Longrightarrow \qquad & \log \mathrm{E}[Z] = \mathrm{E}[\log Z] + \frac{1}{2}\mathrm{Var}[\log Z] \end{aligned}$$

Risk-Free Rate I

- short-term risk-free asset:

$$P_{ft} = \mathrm{E}_t[M_{t+1}] \quad \Longleftrightarrow \quad R_{ft+1} = 1/\mathrm{E}_t[M_{t+1}]$$

$$\Longleftrightarrow \quad \hat{R}_{ft+1} = -\log \mathrm{E}_t[M_{t+1}]$$

$$= \underbrace{-\mathrm{E}_t[\hat{M}_{t+1}] - \frac{\mathrm{Var}_t[\hat{M}_{t+1}]}{2}}_{\text{for log-normal distribution}}$$

Risk-Free Rate II

- long-term risk-free asset:

$$P_{ft}^{t+\tau} = \mathrm{E}_t[M_t^{t+\tau}] \quad \Longleftrightarrow \quad R_{ft}^{t+\tau} = 1/\mathrm{E}_t[M_t^{t+\tau}]$$

$$\Longleftrightarrow \quad \hat{R}_{ft}^{t+\tau} = -\log \mathrm{E}_t[M_t^{t+\tau}]$$

$$\underbrace{= -\sum_{j=1}^{\tau} \mathrm{E}_t[\hat{M}_{t+j}] - \frac{1}{2}\mathrm{Var}_t\left[\sum_{j=1}^{\tau} \hat{M}_{t+j}\right]}_{\text{for log-normal distribution}}$$

Risk-Free Rate III

- Unconditional expectation:

$$\underbrace{\mathrm{E}\left[\frac{1}{R_{ft}}\right] = \mathrm{E}[P_{ft}] = \mathrm{E}[M_t]}_{\text{general case}} \iff \underbrace{\log \mathrm{E}\left[\frac{1}{R_{ft}}\right] = \log \mathrm{E}[M_t] = \mathrm{E}[\hat{M}_t] + \frac{1}{2}\mathrm{Var}[\hat{M}_t]}_{\text{log-normal distribution}}$$

Almost Risk-free Rate

- Projecting 1 on the payoff space from t to $t+\tau$:

$$1 = \underbrace{1_{\mathrm{p}t+\tau}}_{t+\tau \text{ payoff of a portfolio that is self-financing from } t \text{ to } t+\tau} + 1_{\mathrm{p}_\perp t+\tau}$$

- Price:

$$\mathrm{E}_t[M_{\mathrm{p}t+1}] = \mathrm{E}_t[M_{\mathrm{p}t+1}1] = \mathrm{E}_t[M_{\mathrm{p}t+1}1_{\mathrm{p}t+1}] + \overbrace{\mathrm{E}_t[M_{\mathrm{p}t+1}1_{\mathrm{p}_\perp t+1}]}^{=0}$$

$$\Longrightarrow \qquad P_{1_{\mathrm{p}t+1}t} = \mathrm{E}_t[M_{\mathrm{p}t+1}1_{\mathrm{p}t+1}] = \mathrm{E}_t[M_{\mathrm{p}t+1}]$$

- Return:

$$R_{1_\mathrm{p}t+1} = \frac{1_{\mathrm{p}t+1}}{P_{1_{\mathrm{p}t+1}t}} = \frac{1_{\mathrm{p}t+1}}{\mathrm{E}_t[M_{\mathrm{p}t+1}]} = \underbrace{\frac{1}{\mathrm{E}_t[M_{t+1}]} = R_{ft+1}}_{\text{if a risk-free asset exists}}$$

Discounting Future Cash Flows I

- We have:

$$P_t = \overbrace{\mathrm{E}_t[M_{t+1}(P_{t+1} + D_{t+1})]}^{\mathrm{E}_t[(P_{t+1}+D_{t+1})]\mathrm{E}_t[M_{t+1}]+\mathrm{Cov}_t[(P_{t+1}+D_{t+1}),M_{t+1}]}$$

$$\Longrightarrow \qquad P_t = \frac{\mathrm{E}_t[P_{t+1} + D_{t+1}]}{\underbrace{1/\mathrm{E}_t[M_{t+1}]}_{R_{ft+1} \text{ (if risk-free asset exists)}}} + \mathrm{Cov}_t[M_{t+1}, P_{t+1} + D_{t+1}]$$

- Traded discount factor:

$$P_t = \frac{\mathrm{E}_t[P_{t+1} + D_{t+1}]}{\underbrace{R_{pt+1}}_{R_{ft+1} \text{ (if risk-free asset exists)}}} + \mathrm{Cov}_t[M_{pt+1}, P_{t+1} + D_{t+1}]$$

Discounting Future Cash Flows II

- Interpretation of the decomposition

$$P_{t+1} + D_{t+1} = \overbrace{\mathrm{E}_t[P_{t+1} + D_{t+1}]}^{\text{expected payoff}} + \overbrace{\Big((P_{t+1} + D_{t+1}) - \mathrm{E}_t[P_{t+1} + D_{t+1}]\Big)}^{\text{unexpected payoff}}$$

$$\Longrightarrow \qquad P_t = \overbrace{\mathrm{E}_t\Big[M_{t+1}\mathrm{E}_t[P_{t+1} + D_{t+1}]\Big]}^{\text{price of expected payoff}} + \overbrace{\mathrm{E}_t\Big[M_{t+1}\Big((P_{t+1} + D_{t+1}) - \mathrm{E}_t[P_{t+1} + D_{t+1}]\Big)\Big]}^{\text{price of unexpected payoff}}$$

$$= \frac{\mathrm{E}_t[P_{t+1} + D_{t+1}]}{\underbrace{1/\mathrm{E}_t[M_{t+1}]}_{R_{ft+1} \text{ (if risk-free asset exists)}}} + \mathrm{Cov}_t[M_{t+1}, P_{t+1} + D_{t+1}]$$

Discounting Future Cash Flows III

- discounting future dividends:

$$P_t = \sum_{j=1}^{\tau} \underbrace{E_t[M_t^{t+j} D_{t+j}]}_{\underbrace{E_t[M_t^{t+j}]E_t[D_{t+j}]}_{\underbrace{\frac{E_t[D_{t+j}]}{R_{ft}^{t+j}}}_{\text{risk neutral price}}} + \underbrace{Cov_t[M_t^{t+j}, D_{t+j}]}_{\text{risk adjustment}}} + E_t[M_t^{t+\tau} P_{t+\tau}]$$

- if the discounted price $P_{t+\tau}$ converges to zero:

$$\frac{P_t}{D_t} = \sum_{j=1}^{\infty} \left(\frac{E_t[G_{Dt}^{t+j}]}{R_{ft}^{t+j}} + Cov_t[M_t^{t+j}, G_{Dt}^{t+j}] \right)$$

Risk Premium I

- we have for any asset:

$$1 = \underbrace{\mathrm{E}_t[M_t^{t+\tau} R_t^{t+\tau}]}_{\mathrm{Cov}_t[M_t^{t+\tau}, R_t^{t+\tau}] + \underbrace{\mathrm{E}_t[M_t^{t+\tau}]}_{1/R_{ft}^{t+\tau} \text{ if risk-free asset exists}} \mathrm{E}_t[R_t^{t+\tau}]}$$

- linear risk premium

$$\mathrm{E}_t[R_t^{t+\tau}] - \overbrace{\frac{1}{E_t[M_t^{t+\tau}]}}^{R_{ft}^{t+\tau} \text{ if risk-free asset exists}} = -\frac{\mathrm{Cov}_t[M_t^{t+\tau}, R_t^{t+\tau}]}{E_t[M_t^{t+\tau}]}$$

- relative risk premium:

$$\frac{\mathrm{E}_t[R_t^{t+\tau}]}{1/E_t[M_t^{t+\tau}]} - 1 = -\mathrm{Cov}_t[M_t^{t+\tau}, R_t^{t+\tau}]$$

Risk Premium II

- traded discount factor:

$$\underbrace{\frac{\mathrm{E}_t[R_t^{t+\tau}]}{R_{pt}^{t+\tau}} - 1 = -\mathrm{Cov}_t[M_{pt}^{t+\tau}, R_t^{t+\tau}]}_{\text{relative risk premium}} \iff \underbrace{\mathrm{E}_t[R_t^{t+\tau}] - \overbrace{R_{pt}^{t+\tau}}^{R_{ft}^{t+\tau} \text{ if risk-free asset exists}} = -\frac{\mathrm{Cov}_t[M_{pt}^{t+\tau}, R_t^{t+\tau}]}{R_{pt}^{t+\tau}}}_{\text{linear risk premium}}$$

Sharpe Ratio

- Sharpe ratio:

$$\frac{\mathrm{E}_t[R_{t+1}] - 1/E_t[M_{t+1}]}{\sqrt{\mathrm{Var}_t[R_{t+1}]}} = -(1/E_t[M_{t+1}])\mathrm{Corr}_t[M_{t+1}, R_{t+1}]\sqrt{\mathrm{Var}_t[M_{t+1}]}$$

- lower bound for discount factor:

$$\frac{\mathrm{E}_t[R_{t+1}] - 1/E_t[M_{t+1}]}{\sqrt{\mathrm{Var}_t[R_{t+1}]}} \leq \frac{\sqrt{\mathrm{Var}_t[M_{t+1}]}}{\mathrm{E}_t[M_{t+1}]}$$

$$\Longrightarrow \qquad \sqrt{\mathrm{Var}_t[M_{t+1}]} \geq \frac{1}{R_{ft+1}} \frac{\mathrm{E}_t[R_{t+1}] - R_{ft+1}}{\sqrt{\mathrm{Var}_t[R_{t+1}]}}$$

Risk Premium: Lognormal Distribution

- assume $\hat{M}_t = \log M_t$ and $\hat{R}_t = \log R_t$ are jointly normally distributed

$$0 = \mathrm{E}_t[\hat{M}_{t+1}] + \mathrm{E}_t[\hat{R}_{t+1}] + \frac{1}{2} \underbrace{\mathrm{Var}_t[\hat{M}_{t+1} + \hat{R}_{t+1}]}_{\mathrm{Var}_t[\hat{M}_{t+1}] + \mathrm{Var}_t[\hat{R}_{t+1}] + 2\mathrm{Cov}_t[\hat{M}_{t+1}, \hat{R}_{t+1}]}$$

- then:

$$\underbrace{\underbrace{\mathrm{E}_t[\hat{R}_{t+1}] + \frac{1}{2}\mathrm{Var}_t[\hat{R}_{t+1}]}_{\log \mathrm{E}_t[R_{t+1}]} + \underbrace{\mathrm{E}_t[\hat{M}_{t+1}] + \frac{1}{2}\mathrm{Var}_t[\hat{M}_{t+1}]}_{\log \mathrm{E}_t[M_{t+1}]}}_{\log\left(\frac{\mathrm{E}_t[R_{t+1}]}{1/\mathrm{E}_t[M_{t+1}]}\right) = \log\left(\frac{\mathrm{E}_t[R_{t+1}]}{R_{ft+1}}\right) \leftarrow \text{ if } R_{ft+1} \text{ exists}} = -\mathrm{Cov}_t[\hat{M}_{t+1}, \hat{R}_{t+1}]$$

Unconditional Risk Premium

- Unconditional expectation:

$$1 = \mathrm{E}_t[M_t^{t+\tau} R_t^{t+\tau}] \qquad \Longleftrightarrow \qquad 1 = \mathrm{E}[M_t^{t+\tau} R_t^{t+\tau}]$$

- Hence:

$$\mathrm{E}[R_t^{t+\tau}] - \underbrace{\frac{1}{E[M_t^{t+\tau}]}}_{1/\mathrm{E}[P_{ft}^{t+\tau}] \quad \text{if the risk-free asset exists}} = -\frac{\mathrm{Cov}[M_t^{t+\tau}, R_t^{t+\tau}]}{\mathrm{E}[M_t^{t+\tau}]}$$

- Alternatively:

$$0 = \mathrm{E}_t[M_t^{t+\tau}(R_t^{t+\tau} - R_{ft}^{t+\tau})] \qquad \Longleftrightarrow \qquad 0 = \mathrm{E}[M_t^{t+\tau}(R_t^{t+\tau} - R_{ft}^{t+\tau})]$$

- Therefore:

$$\mathrm{E}\left[R_t^{t+\tau} - R_{ft}^{t+\tau}\right] = -\frac{\mathrm{Cov}[M_t^{t+\tau}, R_t^{t+\tau} - R_{ft}^{t+\tau}]}{\mathrm{E}[M_t^{t+\tau}]}$$

Unconditional Sharpe Ratio

- Sharpe ratio:

$$\frac{\mathrm{E}[R_t] - E[R_{ft}]}{\sqrt{\mathrm{Var}[R_t]}} = -\frac{1}{\mathrm{E}[1/R_{ft}]}\mathrm{Corr}[M_t, R_t]\sqrt{\mathrm{Var}[M_t]}$$

- Lower bound is given by:

$$\sqrt{\mathrm{Var}[M_t]} \geq \mathrm{E}\left[\frac{1}{R_{ft}}\right] \frac{\mathrm{E}[R_t] - \mathrm{E}[R_{ft}]}{\sqrt{\mathrm{Var}[R_t]}}$$

- in U.S. data:

  market risk premium ≈ 0.08, volatility ≈ 0.2

Inflation

- Adjusting the present value relation for inflation:

$$\underbrace{P_t^\$}_{\text{price in } t \text{ dollars}} = E_t\Big[\overbrace{M_{t+1}^\$}^{\text{nominal discount factor}}\overbrace{(P_{t+1}^\$ + D_{t+1}^\$)}^{\text{nominal payoff}}\Big] = E_t\left[\overbrace{(M_{t+1}^\$ \times \text{inflation}_{t+1})}^{\text{real discount factor}}\overbrace{\frac{P_{t+1}^\$ + D_{t+1}^\$}{\text{inflation}_{t+1}}}^{\text{payoff in } t \text{ dollars}}\right]$$

$$\iff \underbrace{\frac{P_t^\$}{P_{Ct}^\$}}_{\text{price in consumption bundles}} = E_t\left[(M_{t+1}^\$ \times \text{inflation}_{t+1})\underbrace{\frac{P_{t+1}^\$ + D_{t+1}^\$}{P_{Ct+1}^\$}}_{\text{payoff in consumption bundles}}\right]$$

- Accordingly for returns:

$$1 = E_t[M_{t+1}^C \underbrace{R_{t+1}^C}_{\text{return in consumption bundles}}] = E_t[M_{t+1}^\$ \underbrace{R_{t+1}^\$}_{\text{return in dollars}}]$$

Risk-Free Rate and Inflation

- Risk-free rate in real and nominal terms:

$$\underbrace{R^{C}_{f^{C}\,t+1}}_{\text{real return of an asset with a certain real payout}} = 1/\mathrm{E}_t[M^{C}_{t+1}],$$

$$\underbrace{R^{\$}_{f^{\$}\,t+1}}_{\text{nominal return of an asset with a certain nominal payout}} = 1/\mathrm{E}_t[M^{\$}_{t+1}]$$

- Risk premium in real and nominal terms:

$$\mathrm{E}_t[R^C_{t+1}] - \overbrace{\frac{1}{\mathrm{E}_t[M^C_{t+1}]}}^{R^C_{f^C\,t+1}} = -\mathrm{Cov}_t[M^C_{t+1}, R^C_{t+1}]\frac{1}{\mathrm{E}_t[M^C_{t+1}]}$$

$$\mathrm{E}_t[R^\$_{t+1}] - \underbrace{\frac{1}{\mathrm{E}_t[M^\$_{t+1}]}}_{R^\$_{f^\$\,t+1}} = -\mathrm{Cov}_t[\underbrace{M^\$_{t+1}}_{\frac{M^C_{t+1}}{\mathrm{inflation}_{t+1}}}, R^\$_{t+1}]\frac{1}{\mathrm{E}_t[M^\$_{t+1}]}$$

Risk Premium for the Nominal Risk-free Asset

- risk premium for the inflation-adjusted nominal risk-free rate:

$$\mathrm{E}_t[\overbrace{R^C_{f\$\,t+1}}^{\frac{R^\$_{f\$\,t+1}}{\text{inflation}_{t+1}}}] - \overbrace{\frac{1}{\mathrm{E}_t[M^C_{t+1}]}}^{R^C_{f^C\,t+1}} = -\frac{\mathrm{Cov}_t[M^C_{t+1}, R^C_{f\$\,t+1}]}{\mathrm{E}_t[M^C_{t+1}]}$$

- If inflation_{t+1} uncorrelated with the $M^\$_{t+1}$ and $R^\$_{f\$\,t+1}$:

$$\mathrm{Cov}_t[M^C_{t+1}, R^C_{f\$\,t+1}] = \underbrace{\mathrm{E}_t[\text{inflation}_{t+1}] \times \mathrm{E}_t\left[\frac{1}{\text{inflation}_{t+1}}\right]}_{=1 \text{ if inflation}_{t+1} \text{ known at } t} \mathrm{Cov}_t[M^\$_{t+1}, R^\$_{f\$\,t+1}]$$