Recursive Utility

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Utility Function

- Epstein & Zin (1989):

$$U_t = \left(C_t^{1-\rho} + \delta (E_t U_{t+1}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$

Special Case I: No Uncertainty

$$U_t = \left(C_t^{1-\rho} + \delta U_{t+1}^{1-\rho}\right)^{\frac{1}{1-\rho}} \quad \Longrightarrow \quad U_t^{1-\rho} = C_t^{1-\rho} + \delta \underbrace{U_{t+1}^{1-\rho}}_{C_{t+1}^{1-\rho} + \delta \underbrace{U_{t+2}^{1-\rho}}_{C_{t+2}^{1-\rho} + \delta U_{t+3}^{1-\rho}}} \quad \Longrightarrow \quad U_t = \left(\sum_{j=0}^{\infty} \delta^j C_{t+j}^{1-\rho}\right)^{\frac{1}{1-\rho}}$$

Special Case II: $\rho = \gamma$

$$U_t = \left(C_t^{1-\gamma} + \delta \mathsf{E}_t[U_{t+1}^{1-\gamma}]\right)^{\frac{1}{1-\gamma}} \implies U_t^{1-\gamma} = C_t^{1-\gamma} + \delta \mathsf{E}_t \underbrace{U_{t+1}^{1-\gamma}}_{C_{t+1}^{1-\gamma} + \delta \mathsf{E}_{t+1} \underbrace{U_{t+2}^{1-\gamma}}_{C_{t+2}^{1-\gamma} + \delta \mathsf{E}_{t+2} U_{t+3}^{1-\gamma}}} \implies U_t = \left(\sum_{j=0}^{\infty} \delta^j \mathsf{E}_t[C_{t+j}^{1-\gamma}]\right)^{\frac{1}{1-\gamma}}$$

Elasticity of Intertemporal Substitution

- Derivatives:

$$\frac{\partial U_t}{\partial C_t} = \frac{1}{1-\rho}(C_t^{1-\rho} + \delta U_{t+1}^{1-\rho})^{\frac{1}{1-\rho}-1} \times (1-\rho) C_t^{-\rho}, \qquad \frac{\partial U_t}{\partial C_{t+1}} = \frac{1}{1-\rho}(C_t^{1-\rho} + \delta U_{t+1}^{1-\rho})^{\frac{1}{1-\rho}-1} \times \delta (1-\rho) C_{t+1}^{-\rho}$$

- Hence:

$$\frac{\partial U_t / \partial C_t}{\partial U_t / \partial C_{t+1}} = \frac{1}{\delta} \left(\frac{C_t}{C_{t+1}} \right)^{-\rho} \quad \Longrightarrow \quad \frac{1}{\rho} \left(\log \frac{\partial U_t / \partial C_t}{\partial U_t / \partial C_{t+1}} + \log \delta \right) = \log \frac{C_{t+1}}{C_t} \quad \Longrightarrow \quad \frac{\partial \log \frac{C_{t+1}}{C_t}}{\partial \log \frac{\partial U_t / \partial C_t}{\partial U_t / \partial C_{t+1}}} = \frac{1}{\rho}$$

- FOC:

$$\frac{\partial U_t}{\partial H_{at}} = (1-\rho)C_t^{-\rho}(-P_{at}) + \delta\frac{1-\rho}{1-\gamma}\mathrm{E}_t\Big[\overbrace{U_{t+1}}^{\left(C_{t+1}^{1-\rho}+\delta(E_{t+1}U_{t+2}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}}\right)^{\frac{1-\gamma}{1-\rho}}}{}^{1-\gamma}\Big]^{\frac{1-\rho}{1-\gamma}-1}\mathrm{E}_t\left[\frac{1-\gamma}{1-\rho}\overbrace{U_{t+1}}^{\left(C_{t+1}^{1-\rho}+\delta(E_{t+1}U_{t+2}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}}\right)^{\frac{1-\gamma}{1-\rho}-1}}{}^{\rho-\gamma}(1-\rho)C_{t+1}^{-\rho}(P_{at+1}+D_{at+1})\right] = 0$$

$$\Longleftrightarrow \qquad C_t^{-\rho}(-P_{at}) + \delta\mathrm{E}_t\left[U_{t+1}^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}}\mathrm{E}_t\left[U_{t+1}^{\rho-\gamma}C_{t+1}^{-\rho}(P_{at+1}+D_{at+1})\right] = 0$$

- discount factor:

$$M_{t+1} = \delta\underbrace{\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}}_{\text{power utility}}\underbrace{\left(\frac{\overbrace{U_{t+1}}^{\text{future utility}}}{E_t[U_{t+1}^{1-\gamma}]^{1/(1-\gamma)}}\right)}_{\text{certainty equivalent}}{}^{\rho-\gamma}$$

Discount Factor with Tradable Consumption I

- Assume
$$\frac{U_{t+1}}{W_{t+1}} = \underbrace{\Phi_{t+1}}_{\text{independent of } (\mathbf{H}_t, \mathbf{H}_{t-1}, \dots) \text{ and } (W_t, W_{t-1}, \dots)}$$
- utility:

$$U_t = \left(C_t^{1-\rho} + \delta(E_t[(\Phi_{t+1} \overbrace{W_{t+1}}^{(W_t - C_t)R_{Ht+1}})^{1-\gamma}])^{\frac{1-\rho}{1-\gamma}} \right)^{1/(1-\rho)}$$

$$= \left(C_t^{1-\rho} + \delta(W_t - C_t)^{1-\rho}(E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}])^{\frac{1-\rho}{1-\gamma}} \right)^{1/(1-\rho)}$$

Discount Factor with Tradable Consumption II

- FOC for consumption:

$$\frac{\partial U_t}{\partial C_t}: \qquad C_t^{-\rho} - \delta(W_t - C_t)^{-\rho}\left(E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}]\right)^{\frac{1-\rho}{1-\gamma}} = 0$$

$$\iff \qquad C_t = \delta^{-\frac{1}{\rho}}(W_t - C_t)\left(E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}]\right)^{-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}$$

$$\iff \qquad C_t = \left(\frac{\delta^{-\frac{1}{\rho}}\left(E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}]\right)^{-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}}{1 + \delta^{-\frac{1}{\rho}}\left(E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}]\right)^{-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}}\right) W_t$$

- FOC for portfolio weights:

$$\frac{\partial U_t}{\partial h_{at}}: \qquad E_t\left[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{-\gamma}(R_{at+1} - R_{ft+1})\right] = 0$$

- utility:

$$U_t = \left(C_t^{1-\rho} + \underbrace{\delta(W_t - C_t)^{1-\rho}\overbrace{\left(E_t[\Phi_{t+1}^{1-\gamma} R_{Wt+1}^{1-\gamma}]\right)^{\frac{1-\rho}{1-\gamma}}}^{\frac{C_t^{-\rho}}{\delta(W_t - C_t)^{-\rho}}}}_{(W_t - C_t)C_t^{-\rho}}\right)^{\frac{1}{1-\rho}} = C_t^{\frac{-\rho}{1-\rho}} W_t^{\frac{1}{1-\rho}} = \underbrace{\left(\frac{W_t}{C_t}\right)^{\frac{\rho}{1-\rho}}}_{\Phi_t} W_t$$

Discount Factor with Tradable Consumption III

- Discount factor:

$$M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \Bigg(\underbrace{\frac{\Phi_{t+1} W_{t+1}}{(W_t - C_t) E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}}_{R_{Ht+1} \underbrace{\frac{\Phi_{t+1}}{E_t[\Phi_{t+1}^{1-\gamma} R_{Ht+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}}_{\delta^{\frac{1}{1-\rho}} \underbrace{\frac{(W_{t+1}/C_{t+1})^{\frac{\rho}{1-\rho}}}{(C_t/(W_t - C_t))^{\frac{-\rho}{1-\rho}}}}_{\left(\underbrace{\frac{W_{t+1}}{W_t - C_t}}_{R_{Ht+1}} \right)^{\frac{\rho}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{-\rho}{1-\rho}}}}} \Bigg)^{\rho-\gamma} = \delta^{\frac{1-\gamma}{1-\rho}} G_{Ct+1}^{-\rho \frac{1-\gamma}{1-\rho}} R_{Ht+1}^{\frac{\rho-\gamma}{1-\rho}}$$