Persistent Growth Rates I

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State Variables

- State variable process summarizes history up to time t:

$$\begin{pmatrix} X_{1t} \\ \vdots \\ X_{Nt} \end{pmatrix}, \begin{pmatrix} X_{1t+1} \\ \vdots \\ X_{Nt+1} \end{pmatrix}, \begin{pmatrix} X_{1t+2} \\ \vdots \\ X_{Nt+2} \end{pmatrix}, \ldots$$

- Convenient assumption:

$$\hat{X}_t = \alpha \hat{X}_{t-1} + \sigma \underbrace{\hat{\epsilon}_t}_{(\hat{\epsilon}_{1t} \ldots \hat{\epsilon}_{Kt})}$$

General Specification

$$\hat{G}_{Dt+1} = \hat{\mu}_D + \sum_{n=1}^{N} \phi_{Dn} \overbrace{\hat{X}_{nt}}^{\alpha_{X_n}\hat{X}_{nt-1} + \sigma_{X_n}\hat{\epsilon}_t} + \sigma_D \hat{\epsilon}_{t+1}$$

$$\hat{M}_{t+1} = \hat{\mu}_M + \sum_{n=1}^{N} \phi_{Mn} \hat{X}_{nt} + \sigma_M \hat{\epsilon}_{t+1}$$

Example: Power Utility

Consumption growth:

$$\hat{G}_{Ct+1} = \hat{\mu}_C + \phi_C \hat{X}_t + \sigma_C \hat{\epsilon}_{t+1}$$

Discount factor:

$$\hat{M}_{t+1} = \hat{\delta} - \gamma \hat{G}_{Ct+1} = \underbrace{\hat{\delta} - \gamma \hat{\mu}_C}_{\hat{\mu}_M} + \underbrace{-\gamma \phi_C}_{\phi_M} \hat{X}_t + \underbrace{-\gamma \sigma_C}_{\sigma_M} \hat{\epsilon}_{t+1}$$

Long-term Dividend and Discount Factor

$$\hat{G}_{Dt}^{t+\tau} = \tau\hat{\mu}_D + \sum_{n=1}^{N} \phi_{Dn} \frac{1-\alpha_{X_n}^{\tau}}{1-\alpha_{X_n}} \hat{X}_{nt} + \overbrace{\sum_{j=1}^{\tau} \left(\sigma_D + \sum_{n=1}^{N} \phi_{Dn} \frac{1-\alpha_{X_n}^{\tau-j}}{1-\alpha_{X_n}} \sigma_{X_n} \right) \hat{\epsilon}_{t+j}}^{\text{define: } \hat{Z}_{Dt}^{t+\tau}}$$

$$\hat{M}_{t}^{t+\tau} = \tau\hat{\mu}_M + \sum_{n=1}^{N} \phi_{Mn} \frac{1-\alpha_{X_n}^{\tau}}{1-\alpha_{X_n}} \hat{X}_{nt} + \underbrace{\sum_{j=1}^{\tau} \left(\sigma_M + \sum_{n=1}^{N} \phi_{Mn} \frac{1-\alpha_{X_n}^{\tau-j}}{1-\alpha_{X_n}} \sigma_{X_n} \right) \hat{\epsilon}_{t+j}}_{\text{define: } \hat{Z}_{Mt}^{t+\tau}}$$

Dividend Growth

Unconditional expectation of short-term dividend growth:

$$\mathrm{E}[\hat{G}_{Dt}] = \mu_D$$

Unconditional variance of short-term dividend growth:

$$\mathrm{Var}[\hat{G}_{Dt}] = \sum_n \frac{\phi_{Dn}^2}{1 - \alpha_{X_n}^2} + \sigma_D^2$$

Define fractions of variance coming from state variable n:

$$\omega_{Dn} = \frac{\frac{\phi_{Dn}^2}{1-\alpha_{X_n}^2}}{\sum_n \frac{\phi_{Dn}^2}{1-\alpha_{X_n}^2} + \sigma_D^2}$$

Autocorrelation:

$$\mathrm{Corr}\left[\hat{G}_{Dt}, \hat{G}_{Dt+\tau}\right] = \sum_n \alpha_{X_n}^{\tau} \omega_{Dn}$$

Risk-free Rate

- Short-term:

$$R_{ft+1} = \frac{1}{\mathrm{E}_t[M_{t+1}]} = \frac{1}{\exp\left(\hat{\mu}_M + \sum_n \phi_{Mn}\hat{X}_{nt}\right)} \times \frac{1}{\mathrm{E}\left[\exp\left(\sigma_M \hat{\epsilon}_t\right)\right]}$$

$$\Longrightarrow \quad \hat{R}_{ft+1} = -\left(\hat{\mu}_M + \sum_n \phi_{Mn}\hat{X}_{nt}\right) - \underbrace{\log \mathrm{E}\left[\exp\left(\sigma_M \hat{\epsilon}_t\right)\right]}_{}$$

$$0.5\sigma_M^2 \quad \leftarrow \quad \text{lognormal case}$$

- Long-Term:

$$\hat{R}_{ft}^{t+\tau} = -\mathrm{E}_t[\hat{M}_t^{t+\tau}] - \frac{1}{2}\mathrm{Var}_t[\hat{M}_t^{t+\tau}]$$

$$= -\tau\hat{\mu}_M - \sum_n \phi_{Mn}\frac{1-\alpha_{X_n}^{\tau}}{1-\alpha_{X_n}}\hat{X}_{nt} - \frac{1}{2}\sum_{j=1}^{\tau}\left(\sum_n \phi_{Mn}\frac{1-\alpha_{X_n}^{\tau-j}}{1-\alpha_{X_n}}\sigma_{X_D} + \sigma_M\right)^2$$

Dividend Strip

P-D ratio:

$$\frac{P_{D_{t+\tau}t}}{D_t} = \mathrm{E}_t[M_t^{t+\tau} G_{Dt}^{t+\tau}] = \mathrm{E}_t[e^{\hat{M}_t^{t+\tau} + \hat{G}_{Dt}^{t+\tau}}]$$

$$= e^{(\hat{\mu}_M + \hat{\mu}_D)\tau + \sum_n (\phi_{Dn} + \phi_{Mn}) \frac{1-\alpha_{X_n}^{\tau}}{1-\alpha_{X_n}} \hat{X}_{nt}}$$

$$\times \mathrm{E}_t\left[e^{\sum_{j=1}^{\tau} \left(\sigma_M + \sigma_D + \sum_n (\phi_{Mn} + \phi_{Dn}) \frac{1-\alpha_{X_n}^{\tau-j}}{1-\alpha_{X_n}} \sigma_{X_n}\right) \hat{\epsilon}_{t+j}} \right]$$

Taking logs:

$$\log \frac{P_{D_{t+\tau}t}}{D_t} = \mathrm{E}\left[\log \frac{P_{D_{t+\tau}t}}{D_t}\right] + \sum_n (\phi_{Dn} + \phi_{Mn}) \frac{1-\alpha_{X_n}^{\tau}}{1-\alpha_{X_n}} \hat{X}_{nt}$$