Notation

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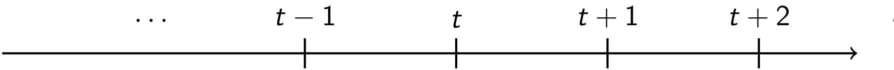
Time

Time is discrete:

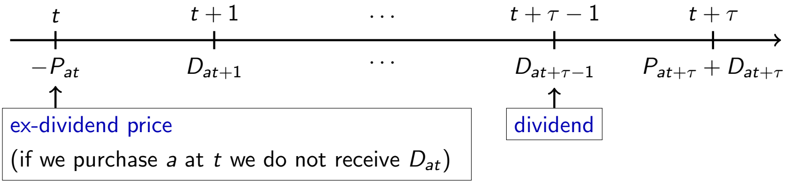


Number of Assets

- countably infinitely many assets
- indexed with $a \in \{1, 2, 3, \dots\}$
- number of assets alive at any given time t is finite

Prices and Dividends

Cash flows:



Sequences of all time t prices and dividends:

$$\boldsymbol{P}_t = (P_{1t}, P_{2t}, P_{3t} \dots), \qquad \boldsymbol{D}_t = (D_{1t}, D_{2t}, D_{3t} \dots)$$

If asset a does not exist at t: $P_{at} = D_{at} = 0$

Dividend Growth Rates

- Dividend growth rate of asset a from t to $t+1$:

$$G_{D_a t+1} = \frac{D_{at+1}}{D_{at}}$$

- multi-period growth rate from time t to $t+\tau$:

$$G_{D_a t}^{t+\tau} = \frac{D_{at+\tau}}{D_{at}} = G_{D_a t+1} \times G_{D_a t+2} \times \cdots \times G_{D_a t+\tau}$$

Portfolios

t — $P_{Ht} = \boldsymbol{H}_t \boldsymbol{P}_t$

$t+1$ — $\boldsymbol{H}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1})$; $P_{Ht+1} = \boldsymbol{H}_{t+1}\boldsymbol{P}_{t+1}$

$t+2$ — $\boldsymbol{H}_{t+1}(\boldsymbol{P}_{t+2} + \boldsymbol{D}_{t+2})$; $P_{Ht+2} = \boldsymbol{H}_{t+2}\boldsymbol{P}_{t+2}$

portfolio holdings

$\boldsymbol{H}_t = (H_{1t}, H_{2t}, H_{3t} \dots)$

If a does not exist at t then $H_{at} = 0$.

portfolio weights

$$h_{at} = \frac{H_{at}P_{at}}{\boldsymbol{H}_t\boldsymbol{P}_t}, \qquad \boldsymbol{h}_t = (h_{1t}, h_{2t}, \dots)$$

An equal weighted portfolio has identical weights for all assets: $h_{at} = h_{bt}$ for all a, b.

portfolio value

$$P_{Ht} = \boldsymbol{H}_t\boldsymbol{P}_t = \sum_a H_{at}P_{at} = \lim_{A\to\infty}\sum_{a=1}^{A} H_{at}P_{at}$$

(since $\boldsymbol{H}_t$ contains only finitely many non-zero elements there exists $k > 0$ such that $H_{at} = 0$ for all $a > k$).

portfolio dividend

$$D_{Ht+1} = \boldsymbol{H}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1}) - P_{Ht+1}$$
$$= \underbrace{(\boldsymbol{H}_t - \boldsymbol{H}_{t+1})}_{\text{trade}}\boldsymbol{P}_{t+1} + \underbrace{\boldsymbol{H}_t\boldsymbol{D}_{t+1}}_{\text{portfolio dividend if } \boldsymbol{H}_t \text{ constant}}$$

If $D_{Ht} \neq 0$, the portfolio has inflows or outflows. If $D_{Ht} = 0$ from time t to $t+\tau$ we say the portfolio is self-financing during this time period.

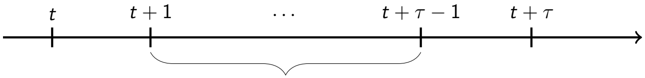
portfolio payoff:

$\boldsymbol{H}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1}) = \sum_a H_{at}(P_{at+1} + D_{at+1})$

Short Selling I

	t	$t+1$	$t+2$	$\cdots$	$t+\tau$
long:	$-P_{at}$	$+D_{at+1}$	$+D_{at+2}$		$+P_{at+\tau} + D_{at+\tau}$
short:	$+P_{at}$	$-D_{at+1}$	$-D_{at+2}$		$-P_{at+\tau} - D_{at+\tau}$

Short Selling II



	asset	dividends	asset
original owners	lend out	receive	retrieve
	↓	↑	↑
short seller	borrow & sell	pay	purchase & return
	↓		↑
other investors	purchase	receive (from asset)	sell

Returns: Single-Period

- Return from time t to $t+1$:

$$R_{at+1} = \frac{P_{at+1} + D_{at+1}}{P_{at}}, \qquad \boldsymbol{R}_{t+1} = (R_{1t+1}, R_{2t+1}, \dots)$$

- portfolio return

$$R_{Ht+1} = \frac{\boldsymbol{H}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1})}{\boldsymbol{H}_t \boldsymbol{P}_t} = \sum_a \frac{H_{at}\overbrace{(P_{at+1} + D_{at+1})}^{P_{at} R_{at+1}}}{P_{Ht}} = \sum_a \underbrace{\frac{H_{at} P_{at}}{P_{Ht}}}_{h_{at}} R_{at+1}$$

- or:

$$R_{Ht+1} = \frac{\boldsymbol{H}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1})}{\boldsymbol{H}_t \boldsymbol{P}_t} = \frac{P_{Ht+1} + D_{Ht+1}}{P_{Ht}}$$

- constant portfolio holdings:

$$R_{Ht+1} = \frac{P_{Ht+1} + \boldsymbol{H}_t \boldsymbol{D}_{t+1}}{P_{Ht}}$$

Returns: Multi-Period

- multi-period (or compound) return:

$$R_{at}^{t+\tau} = R_{at+1} \times R_{at+2} \times \cdots \times R_{at+\tau}$$

	t	$t+1$	$\cdots$	$t+\tau-1$	$t+\tau$
cash flow:	-1	0	$\cdots$	0	$R_{at}^{t+\tau}$
portfolio holdings for asset a:	$1/P_{at}$	$1/P_{at} \times R_{at+1}$		$1/P_{at+\tau-1} \times R_{at}^{t+\tau-1}$	
(zero holdings for all other assets)					

Risk-Free Asset

- generic risk-free zero coupon bond:

t	$t+1$	$t+2$	$\cdots$	$t+\tau$
$-P_{ft}^{t+\tau}$	0	0	$\cdots$	1

- τ-period risk-free rate

$$R_{ft}^{t+\tau} = 1/P_{ft}^{t+\tau}, \qquad \text{if } \tau = 1\text{: } R_{ft+1}$$

- yield to maturity

$$\left(R_{ft}^{t+\tau}\right)^{1/\tau} = \left(1/P_{ft}^{t+\tau}\right)^{1/\tau}$$

- return for time periods less than maturity

$$P_{ft+j}^{t+\tau}/P_{ft}^{t+\tau}, \qquad j \in [1,\tau),\ \tau > 1$$

Market Portfolio: Total Market Value

- number of shares outstanding of assets a:

$$\overline{H}_{at}$$

- total market value or market capitalization of a:

$$\overline{P}_{at} = \overline{H}_{at} P_{at}.$$

Market Portfolio: Market Weights

- total value of all securities in the market:

$$\overline{P}_{mt} = \overline{\boldsymbol{H}}_t \boldsymbol{P}_t$$

- aggregate dividend: $\overline{\boldsymbol{H}}_{t-1} \boldsymbol{D}_t$
- Market weights:

$$h_{mat} = \frac{\overline{H}_{at} P_{at}}{\overline{\boldsymbol{H}}_t \boldsymbol{P}_t}, \qquad \boldsymbol{h}_t = (h_{1t}, h_{2t}, \dots)$$

- Return of the market portfolio:

$$R_{mt+1} = \boldsymbol{h}_{mt} \boldsymbol{R}_{t+1}$$

- in general

$$R_{mt+1} \neq \frac{\overline{P}_{mt+1} + \overline{\boldsymbol{H}}_t \boldsymbol{D}_{t+1}}{\overline{P}_{mt}}$$

Risk Premium

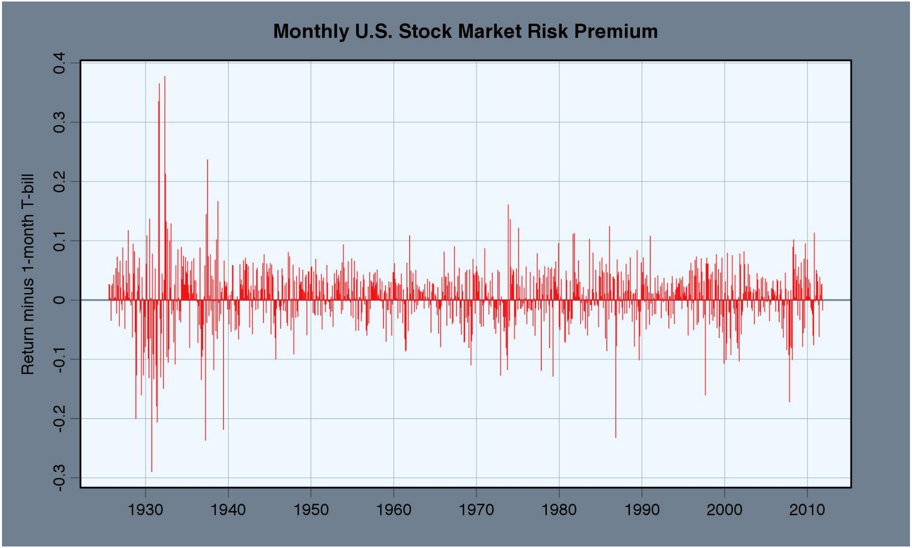
- (Linear) risk premium for time horizon τ:

$$E_t[R_t^{t+\tau}] - R_{ft}^{t+\tau}.$$

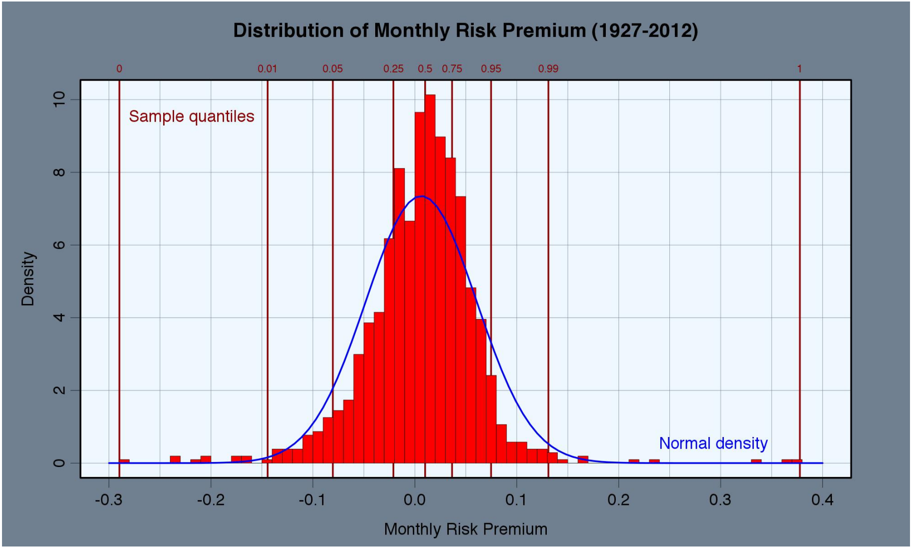
- Sometimes we are also interested in the relative risk premium

$$E_t[R_t^{t+\tau}]/R_{ft}^{t+\tau}.$$

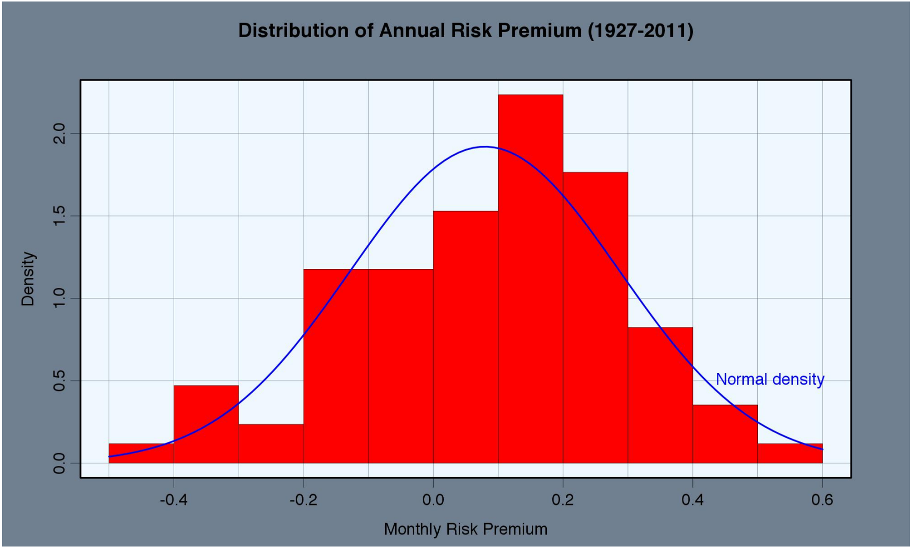
Monthly U.S. Risk Premium



Historical Distribution of U.S. Risk Premium



Distribution of U.S. Risk Premium: Annual



Summary Statistics for Historical Risk Premium

	Monthly	Annual
Mean	0.00626	0.0794
Volatility	0.0543	0.208
Standard deviation of the mean	0.00169	0.0225
Skewness	0.17	-0.29
Kurtosis	7.29	-0.22
Number of observations	1036	85

Sharpe Ratio

Sharpe ratio for time horizon τ:

$$\frac{\mathrm{E}_t[R_t^{t+\tau}] - R_{ft}^{t+\tau}}{\mathrm{SD}[R_t^{t+\tau}]}$$

Historical Sharpe Ratio

- Monthly:

$$\text{Sharpe ratio} = \frac{\text{risk premium}}{\text{standard deviation}} = \frac{0.0626}{0.0543} = 0.12$$

- Annually:

$$\text{Sharpe ratio} = \frac{\text{risk premium}}{\text{standard deviation}} = \frac{0.0794}{0.208} = 0.38$$