Mean-Variance Analysis

Jan Schneider

McCombs School of Business
University of Texas at Austin

Beta Representation of the Risk Premium

- risk premium

$$E_t[R_t^{t+\tau}] - \underbrace{R_{1_p t}^{t+\tau}}_{= R_{ft}^{t+\tau} \text{ if a risk-free asset exists}} = -\frac{\text{Cov}_t[R_t^{t+\tau}, M_{pt}^{t+\tau}]}{E_t[M_{pt}^{t+\tau}]} = \overbrace{-\frac{\text{Cov}_t[R_t^{t+\tau}, R_{Mt}^{t+\tau}]}{E_t[R_{Mt}^{t+\tau}]}}^{\text{divide by the price of } M_{pt}^{t+\tau}}$$

- this equation holds also for the traded discount factor:

$$E_t[R_{Mt}^{t+\tau}] - R_{1_p t}^{t+\tau} = -\frac{\text{Cov}_t[R_{Mt}^{t+\tau}, R_{Mt}^{t+\tau}]}{E_t[R_{Mt}^{t+\tau}]}$$

$$\Rightarrow \qquad E_t[R_{Mt}^{t+\tau}] = -\frac{\text{Var}_t[R_{Mt}^{t+\tau}]}{E_t[R_{Mt}^{t+\tau}] - R_{1_p t}^{t+\tau}}$$

- plug into equation above:

$$E_t[R_{t+1}] - R_{1_p t}^{t+\tau} = \frac{\text{Cov}_t[R_{t+1}, R_{Mt}^{t+\tau}]}{\text{Var}_t[R_{Mt}^{t+\tau}]}\left(E_t[R_{Mt}^{t+\tau}] - R_{1_p t}^{t+\tau}\right)$$

Unconditional Beta Representation

$$\mathrm{E}[R_{t+1}] - \underbrace{\frac{1}{\mathrm{E}[1/R_{1_p t}^{t+\tau}]}}_{\frac{1}{\mathrm{E}[P_{ft}]}} = \frac{\mathrm{Cov}[R_{t+1}, R_{Mt+1}]}{\mathrm{Var}[R_{Mt+1}]} \left(\mathrm{E}[R_{Mt+1}] - \frac{1}{\mathrm{E}[1/R_{1_p t}^{t+\tau}]} \right)$$

$\frac{1}{\mathrm{E}[P_{ft}]}$ $\leftarrow$ if the risk-free asset exists

Maximum Sharpe Ratio

$$\frac{\mathrm{E}_t[R_t^{t+\tau}] - R_{1_pt}^{t+\tau}}{\sqrt{\mathrm{Var}_t[R_t^{t+\tau}]}} = -R_{1_pt}^{t+\tau}\mathrm{Corr}_t[M_{pt}^{t+\tau}, R_t^{t+\tau}]\sqrt{\mathrm{Var}_t[M_{pt}^{t+\tau}]}$$

$$\Longrightarrow \frac{\mathrm{E}_t[R_t^{t+\tau}] - \overbrace{R_{1_pt}^{t+\tau}}^{R_{ft}^{1+\tau} \text{ if a risk-free asset exists}}}{\sqrt{\mathrm{Var}_t[R_t^{t+\tau}]}} \leq R_{1_pt}^{t+\tau}\sqrt{\mathrm{Var}_t[M_{pt}^{t+\tau}]} \underbrace{\leq}_{\text{since } M_t^{t+\tau} = M_{pt}^{t+\tau} + M_{p_\perp t}^{t+\tau}} R_{1_pt}^{t+\tau}\sqrt{\mathrm{Var}_t[M_{t+1}]}$$

Mean-Variance Efficient Frontier I

- $H_t \dots H_{t+\tau-1}$ self-financing portfolio
- portfolio payoff $P_{Ht+\tau} + D_{Ht+\tau}$ is mean-variance efficient q_t to $t+\tau$ if

$$\left.\begin{array}{c} P_H(q_t) = P_{H'}(q_t) \\ \mathrm{E}[P_{Ht+\tau} + D_{Ht+\tau}|q_t] = \mathrm{E}[P_{H'\,t+\tau} + D_{H'\,t+\tau}|q_t] \end{array}\right\} \Longrightarrow \begin{array}{r} \mathrm{Var}[P_{Ht+\tau} + D_{Ht+\tau}|q_t] \\ \leq \mathrm{Var}[P_{H'\,t+\tau} + D_{H'\,t+\tau}|q_t] \end{array}$$

Mean-Variance Efficient Frontier II

- Suppose H is mean-variance efficient. Suppose H' satisfies conditions on the previous slide. Then:

$$\mathrm{E}[R_{H't}^{t+\tau}|q_t] = \mathrm{E}\left[\left.\frac{P_{H't+\tau} + D_{H't+\tau}}{P_{H'}(q_t)}\right| q_t\right] = \mathrm{E}\left[\left.\frac{P_{Ht+\tau} + D_{Ht+\tau}}{P_H(q_t)}\right| q_t\right] = \mathrm{E}[R_{Ht}^{t+\tau}|q_t]$$

$$\text{and}\quad \mathrm{Var}[R_{Ht}^{t+\tau}|q_t] = \mathrm{Var}\left[\left.\frac{P_{Ht+\tau} + D_{Ht+\tau}}{P_H(q_t)}\right| q_t\right] \leq \mathrm{Var}\left[\left.\frac{P_{H't+\tau} + D_{H't+\tau}}{P_{H'}(q_t)}\right| q_t\right] = \mathrm{Var}[R_{H't}^{t+\tau}|q_t]$$

- we have:

mean-variance efficiency in payoffs $\Leftrightarrow$ mean-variance efficiency in returns

Portfolios of the Discount Factor and the Unity Payoff I

- Consider the market of all self-financing portfolios between q_t and $t + \tau$
- Define

$$F_{q_t}^{t+\tau} = \left\{ Y : Y = aM_{pt}^{t+\tau} + b1_{pt}^{t+\tau} \text{ for some } a, b \in \mathbb{R} \right\}$$

- Consider an arbitrary traded payoff Y. Projecting Y on F:

$$Y = Y_F + Y_{F_\perp}$$

- some properties of Y:

1. $E_t[Y] = E_t[Y_F] \qquad \longleftarrow \qquad E_t[Y_{F_\perp}] = \overbrace{E_t[Y_{F_\perp} 1^{t+\tau}_{p_\perp t}]}^{=0} + \overbrace{E_t[Y_{F_\perp} 1^{t+\tau}_{pt}]}^{=0} = 0$

2. $\mathrm{Cov}_t[Y_F, Y_{F_\perp}] = 0 \qquad \longleftarrow \qquad \mathrm{Cov}_t[Y_F, Y_{F_\perp}] = \underbrace{E_t[Y_F Y_{F_\perp}]}_{=0} - E_t[Y_F] \underbrace{E_t[Y_{F_\perp}]}_{=0} = 0$

3. $Y_{F_\perp} \neq 0 \implies \mathrm{Var}_t[Y_{F_\perp}] > 0 \qquad \longleftarrow \qquad \mathrm{Var}_t[Y_{F_\perp}] = E_t[Y^2_{F_\perp}] > 0 \quad \text{if } Y_{F_\perp} \neq 0$

4. $P_{Yt} = P_{Y_F t} \qquad \longleftarrow \qquad P_{Yt} = E_t[M^{t+\tau}_{pt}(Y_F + Y_{F_\perp})] = E_t[M^{t+\tau}_{pt} Y_F] = P_{Y_F t}$

- corresponding decomposition for returns:

$$R^{t+\tau}_{Yt} = \frac{Y}{P_{Yt}} = \frac{Y_F + Y_{F_\perp}}{P_{Yt}} = \underbrace{\frac{Y_F}{P_{Y_F t}}}_{R_{Y_F t+1}} + \underbrace{\frac{Y_{F_\perp}}{P_{Yt}}}_{z_{F_\perp}}$$

Portfolios of the Discount Factor and the Unity Payoff III

- All payoff in F are mean variance efficient.
  Proof. Choose an arbitrary traded payoff Y. Then we can decompose Y as on the previous slide. Since $P_Y = P_{Y_F}$ and since $Y_{F_\perp}$ only adds noise, Y cannot be mean-variance efficient if $Y_{F_\perp} \neq 0$

Portfolios of the Discount Factor and the Unity Payoff IV

- All payoff in E_{q_t} are mean variance efficient.
  Proof.Choose $Y \in F$. There exist a m-v efficient traded payoff Y' such that

$$P_{Yt} = P_{Y't} \quad \text{and} \quad \mathrm{E}_t[Y] = \mathrm{E}_t[Y']$$

  By the argument above Y' is in F. Hence $Y - Y' \in F$. But $Y - Y'$ is also orthogonal to F:

$$\mathrm{E}_t[(Y - Y')1_{pt}^{t+\tau}] + \mathrm{E}_t[(Y - Y')1_{p_\perp t}^{t+\tau}] = \mathrm{E}_t[Y - Y'] = 0$$
$$\mathrm{E}_t[M_{pt}^{t+\tau}(Y - Y')] = P_{Yt} - P_{Y't} = 0$$

  Therefore, since $Y - Y' \in F$ and $Y - Y' \notin F$, $Y - Y' = 0$.
- Hence we have for any market between q_t and $t + \tau$:

$$Y \text{ is mean-variance efficient} \quad \Longleftrightarrow \quad Y \in F_{q_t}^{t+\tau}$$

2 M-V Efficient Portfolios $\longrightarrow$ M-V Frontier

- any mean-variance efficient return is given by

$$R_{ht}^{t+\tau} = hR_{Mt}^{t+\tau} + (1-h)R_{1t}^{t+\tau} = R_{1t}^{t+\tau} + h(R_{Mt}^{t+\tau} - R_{1t}^{t+\tau})$$

Minimum-Variance Portfolio

- variance of the return on the previous slide:

$$\text{Var}_t[R_{ht}^{t+\tau}] = \text{Var}_t[R_{1t}^{t+\tau}] + h^2 \text{Var}_t[R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}] + 2h\text{Cov}_t[R_{1t}^{t+\tau}, R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}]$$

- minimum of the variance:

$$\frac{\partial \text{Var}_t[R_{ht}^{t+\tau}]}{\partial h} = 0 \quad \Longrightarrow \quad h = -\frac{\text{Cov}_t[R_{1t}^{t+\tau}, R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}]}{\text{Var}_t[R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}]}$$

If a risk-free asset is traded, then $h = 0$ and $R_{ht}^{t+\tau} = R_{ft}^{t+\tau}$.

Zero-Covariance Portfolio

- Covariance between two mean-variance efficient returns R_{h_1} and R_{h_2}:

$$\text{Cov}_t[R_{h_1 t}^{t+\tau}, R_{h_2 t}^{t+\tau}] = \text{Var}_t[R_{1t}^{t+\tau}] + h_1 h_2 \text{Var}_t[R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}] + (h_1 + h_2)\text{Cov}_t[R_{1t}^{t+\tau}, R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}]$$

- hence: $\text{Cov}_t[R_{h_1 t}^{t+\tau}, R_{h_2 t}^{t+\tau}] = 0 \iff$

$$h_2 = -\frac{\text{Var}_t[R_{1t}^{t+\tau}] + h_1 \text{Cov}_t[R_{1t}^{t+\tau}, R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}]}{h_1 \text{Var}_t[R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}] + \text{Cov}_t[R_{1t}^{t+\tau}, R_{Mt}^{t+\tau} - R_{1t}^{t+\tau}]}$$

M-V Efficiency and Beta Representation

- Choose any traded return $R_t^{t+\tau}$:

$$\overbrace{R_{h_1 t}^{t+\tau} + \underbrace{\beta_t}_{\in \mathbb{R}}(R_{h_2 t}^{t+\tau} - R_{h_1 t}^{t+\tau})}$$

$$R_t^{t+\tau} = R_{Ft}^{t+\tau} + z_{F_\perp}$$

$$\Longrightarrow \begin{cases} \mathrm{E}_t[R_t^{t+\tau}] = \mathrm{E}_t[R_{h_1 t}^{t+\tau}] + \beta_t \left(\mathrm{E}_t[R_{h_2 t}^{t+\tau}] - \mathrm{E}_t[R_{h_1 t}^{t+\tau}]\right) \\ \mathrm{Cov}_t[R_t^{t+\tau}, R_{h_2 t}^{t+\tau}] = \beta_t \mathrm{Var}_t[R_{h_2 t}^{t+\tau}] \implies \beta_t = \dfrac{\mathrm{Cov}_t[R_t^{t+\tau}, R_{h_2 t}^{t+\tau}]}{\mathrm{Var}_t[R_{h_2 t}^{t+\tau}]} \end{cases}$$

- hence:

$$\mathrm{E}_t[R_t^{t+\tau}] = \underbrace{\mathrm{E}_t[R_{h_1 t}^{t+\tau}]}_{\text{zero-covariance portfolio, } = R_{ft+1} \text{ if risk-free asset exists}} + \frac{\mathrm{Cov}_t[R_t^{t+\tau}, R_{h_2 t}^{t+\tau}]}{\mathrm{Var}_t[R_{h_2 t}^{t+\tau}]} \left(\mathrm{E}_t[R_{h_2 t}^{t+\tau}] - \mathrm{E}_t[R_{h_1 t}^{t+\tau}]\right)$$

Market Portfolio M-V Efficient $\Longrightarrow$ CAPM

- Suppose a risk-free asset exists and suppose the market portfolio is mean-variance efficient. Then we have:

$$\mathrm{E}_t[R_t^{t+\tau}] = R_{ft}^{t+\tau} + \frac{\mathrm{Cov}_t[R_t^{t+\tau}, R_{mt}^{t+\tau}]}{\mathrm{Var}_t[R_{mt}^{t+\tau}]}\left(\mathrm{E}_t[R_{mt}^{t+\tau}] - R_{ft}^{t+\tau}\right)$$

  This model of expected returns is known as the capital asset pricing model (CAPM).

- Suppose $R^{t+\tau}_{h_1 t}$ and $R^{t+\tau}_{h_2 t}$ are mean-variance efficient
- suppose $\mathrm{Cov}_t[R^{t+\tau}_{h_1 t}, R^{t+\tau}_{h_2 t}] = 0$
- Then:

$$M^{t+\tau}_t = \frac{1}{\mathrm{E}_t[R^{t+\tau}_{h_1 t}]} - (R^{t+\tau}_{h_2 t} - \mathrm{E}_t[R^{t+\tau}_{h_2 t}])\frac{\mathrm{E}_t[R^{t+\tau}_{h_2 t}] - \mathrm{E}_t[R^{t+\tau}_{h_1 t}]}{\mathrm{E}_t[R^{t+\tau}_{h_1 t}]\mathrm{Var}_t[R^{t+\tau}_{h_2 t}]}$$

- Proof:

$$\mathrm{E}_t[M^{t+\tau}_t \overbrace{R^{t+\tau}_t}^{\text{any return}}] = \frac{\mathrm{E}_t[R^{t+\tau}_t]}{\mathrm{E}_t[R^{t+\tau}_{h_1 t}]} - \Big(\overbrace{\mathrm{E}_t[R^{t+\tau}_t R^{t+\tau}_{h_2 t}] - \mathrm{E}_t[R^{t+\tau}_t]\mathrm{E}_t[R^{t+\tau}_{h_2 t}]}^{\mathrm{Cov}_t[R^{t+\tau}_t, R^{t+\tau}_{h_2 t}]}\Big)\frac{\mathrm{E}_t[R^{t+\tau}_{h_2 t}] - \mathrm{E}_t[R^{t+\tau}_{h_1 t}]}{\mathrm{E}_t[R^{t+\tau}_{h_1 t}]\mathrm{Var}_t[R^{t+\tau}_{h_2 t}]}$$

$$= 1$$

- For example, if the CAPM holds:

$$M_t^{t+\tau} = \frac{1}{R_{ft}^{t+\tau}} - \left(R_{mt}^{t+\tau} - R_{ft}^{t+\tau}\right) \frac{\mathrm{E}_t[R_{mt}^{t+\tau}] - R_{ft}^{t+\tau}}{R_{ft}^{t+\tau}\mathrm{Var}_t[R_{mt}^{t+\tau}]}$$

Maximizing Sharpe Ratio

- Sharpe ratio of a portfolio H:

$$\frac{E_t[R_{Ht+1}] - R_{ft+1}}{SD_t[R_{Ht+1}]} = \frac{\sum_{a=1}^{A} \overbrace{h_a(E_t[R_{at+1}] - R_{ft+1})}^{\text{since: } \sum h_a = 1}}{\sqrt{\sum_{a=1}^{A} \sum_{b=1}^{A} h_a h_b \text{Cov}[R_{at+1}, R_{bt+1}]}}$$

- Derivative:

$$\frac{\partial(\text{Sharpe ratio})}{\partial h_a} = \frac{(E_t[R_{at+1}] - R_{ft+1})SD_t[R_{Ht+1}] - (E[R_{Ht+1}] - R_{ft+1})\frac{1}{2}\text{Var}[R_{Ht+1}]^{-\frac{1}{2}} 2 \overbrace{\sum_b h_b \text{Cov}_t[R_{at+1}, R_{bt+1}]}^{\text{Cov}_t\left[R_{at+1}, \overbrace{\sum_b h_b R_{bt+1}}^{R_{Ht+1}}\right]}}{\text{Var}_t[R_{Ht+1}]}$$

$$= \frac{E_t[R_{at+1}] - R_{ft+1} - (E_t[R_{Ht+1}] - R_{ft+1})\text{Var}_t[R_{Ht+1}]^{-1}\text{Cov}_t[R_{at+1}, R_{Ht+1}]}{\text{Var}_t[R_{Ht+1}]^{\frac{1}{2}}}$$