Log-Normal IID World

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Exponential Growth

- Consider a random walk with drift:

$$Y_{t+1} = Y_t + \mu + \epsilon_{t+1}$$

$$\Longrightarrow \quad \text{growth rate:} \qquad G_{Yt+1} = \frac{Y_{t+1}}{Y_t} = 1 + \frac{\mu}{Y_t} + \frac{\epsilon_{t+1}}{Y_t}$$

- We can model growth rates independent of levels if model growth linear in logs:

$$Y_{t+1} = Y_t \mu \epsilon_{t+1} \qquad \Longleftrightarrow \qquad \hat{Y}_{t+1} = \hat{Y}_t + \hat{\mu} + \hat{\epsilon}_{t+1}$$

$$\Longrightarrow \quad \text{growth rate:} \quad G_{Yt+1} = \frac{Y_{t+1}}{Y_t} = \mu \epsilon_{t+1} \qquad \Longleftrightarrow \qquad \hat{G}_{Yt+1} = \hat{\mu} + \hat{\epsilon}_{t+1}$$

Base Variables

- iid shocks:

$$\epsilon_t = \begin{pmatrix} \epsilon_{1t} \\ \vdots \\ \epsilon_{Kt} \end{pmatrix}, \epsilon_{t+1} = \begin{pmatrix} \epsilon_{1t+1} \\ \vdots \\ \epsilon_{Kt+1} \end{pmatrix}, \epsilon_{t+2} = \begin{pmatrix} \epsilon_{1t+2} \\ \vdots \\ \epsilon_{Kt+2} \end{pmatrix}, \ldots$$

- $\mathrm{E}[\hat{\epsilon}_{kt}] = 0$, $\mathrm{Var}[\hat{\epsilon}_{kt}] = 1$
- linear combination of base variables:

$$\sigma\hat{\epsilon}_t = \sigma_1\hat{\epsilon}_{1t} + \cdots + \sigma_K\hat{\epsilon}_{Kt}$$

$\Longrightarrow$ Expectation: $$\mathrm{E}_t[\sigma\hat{e}_{t+\tau}] = 0$$

Variance: $$\mathrm{Var}_t[\sigma\hat{e}_{t+\tau}] = \sigma_1^2 + \cdots + \sigma_K^2 = \sigma^2$$

Covariance: $$\mathrm{Cov}_t[\sigma_a\hat{e}_{t+\tau}, \sigma_b\hat{e}_{t+\tau}] = \sum_{j=1}^{K}\sigma_{aj}\sigma_{bj} = \sigma_a\sigma_b$$

General Specification for Discount Factor, Dividend Growth

- Suppose

$$\hat{G}_{Dt} = \hat{\mu}_D + \sigma_D \hat{\epsilon}_t$$
$$\hat{M}_t = \hat{\mu}_M + \sigma_M \hat{\epsilon}_t$$

Term Structure of the Risk-Free Rate

- Short-term risk-free rate:

$$R_{ft+1} = \frac{1}{\mathrm{E}_t[M_{t+1}]}$$

- Long-term rate from t to $t+\tau$:

$$R_{ft}^{t+\tau} = \frac{1}{\mathrm{E}_t[M_{t+1} \cdots M_{t+\tau}]}$$

- Hence if the discount factor is iid the term structure is flat:

$$R_{ft}^{t+\tau} = \frac{1}{\mathrm{E}_t[M_{t+1} \cdots M_{t+\tau}]} = \frac{1}{\mathrm{E}_t[M_{t+1}]} \cdots \frac{1}{\mathrm{E}_t[M_{t+\tau}]} = \frac{1}{\mathrm{E}_t[M_{t+1}]^{\tau}} = (R_{ft+1})^{\tau}$$

- Log-normal case:

$$\hat{R}_{ft}^{t+\tau} = -\mathrm{E}_t[\hat{M}_t^{t+\tau}] - 0.5\mathrm{Var}_t[\hat{M}_t^{t+\tau}] = -\tau\left(\hat{\mu}_M + 0.5\sigma_M^2\right)$$

General Assets

- Price/dividend ratios:

$$\frac{P_t}{D_t} = \sum_{\tau=1}^{\infty} \mathrm{E}_t[M_t^{t+\tau} G_{Dt}^{t+\tau}] = \sum_{\tau=1}^{\infty} \mathrm{E}_t[M_{t+1} G_{Dt+1}]^{\tau} = \frac{\mathrm{E}[M_{t+1} G_{Dt+1}]}{1 - \mathrm{E}[M_{t+1} G_{Dt+1}]}$$

- Return:

$$\hat{R}_{t+1} = \log \frac{P_{t+1} + D_{t+1}}{P_t} = \log \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) - \log \left(\frac{P_t}{D_t}\right) + \hat{G}_{Dt+1}$$

- Risk premium with log-normal distribution:

$$-\mathrm{Cov}_t[\hat{M}_{t+1}, \hat{R}_{t+1}] = -\mathrm{Cov}_t[\hat{M}_{t+1}, \hat{G}_{Dt+1}] = -\sigma_M \sigma_D$$

Duration

$$\sum_{j=1}^{\tau} \mathrm{E}[M_t G_{Dt}]^j = \frac{\mathrm{E}[M_{t+1} G_{Dt+1}] - \mathrm{E}[M_{t+1} G_{Dt+1}]^{\tau+1}}{1 - \mathrm{E}[M_{t+1} G_{Dt+1}]}$$

$$\Longrightarrow \underbrace{\frac{D_t \sum_{j=1}^{\tau} \mathrm{E}[M_t G_{Dt}]^j}{P_t}}_{\text{fraction of } P_t \text{ due to dividends paid over first } \tau \text{ periods}} = \frac{\frac{\mathrm{E}[M_{t+1} G_{Dt+1}] - \mathrm{E}[M_{t+1} G_{Dt+1}]^{\tau+1}}{1 - \mathrm{E}[M_{t+1} G_{Dt+1}]}}{\frac{\mathrm{E}[M_{t+1} G_{Dt+1}]}{1 - \mathrm{E}[M_{t+1} G_{Dt+1}]}} = 1 - \mathrm{E}[M_{t+1} G_{Dt+1}]^{\tau}$$

$$= 1 - \left(\frac{P_t / D_t}{1 + P_t / D_t} \right)^{\tau}$$

$$\Longrightarrow \tau = \frac{\log \left[1 - \frac{D_t \sum_{j=1}^{\tau} \mathrm{E}[M_t G_{Dt}]^j}{P_t} \right]}{\log \left[\frac{P_t / D_t}{1 + P_t / D_t} \right]}$$

Power Utility

- Suppose:

$$\hat{G}_{Ct+1} = \hat{\mu}_C + \sigma_C \hat{\epsilon}_{t+1}$$

- Discount factor:

$$\hat{M}_{t+1} = \hat{\delta} - \gamma \hat{G}_{Ct+1} = \hat{\delta} - \gamma(\hat{\mu}_C + \sigma_C \hat{\epsilon}_{t+1})$$

- Risk-free rate for the log-normal case:

$$\hat{R}_{ft+1} = -\mathrm{E}_t[\hat{M}_{t+1}] - 0.5\mathrm{Var}_t[\hat{M}_{t+1}] = -\delta + \gamma \hat{\mu}_C - 0.5\gamma^2 \sigma_C^2$$

- Risk premium for the log-normal case:

$$-\mathrm{Cov}_t[\hat{M}_{t+1}, \hat{R}_{t+1}] = \gamma \sigma_C \sigma_D$$

Quadratic Utility

- we have

$$E_t[R_{t+1}] - R_{ft+1} = \frac{\text{Cov}_t\left[R_{t+1}, \frac{u'_{t+1}(C_{t+1})}{u'_t(C_t)}\right]}{\text{Cov}_t\left[R_{mt+1}, \frac{u'_{t+1}(C_{t+1})}{u'_t(C_t)}\right]}(E_t[R_{mt+1}] - R_{ft+1})$$

- suppose $u_{t+j}(C_{t+j})' = \delta^j(1 - bC_{t+j})$:

$$E_t[R_{t+1}] - R_{ft+1} = \frac{\text{Cov}_t\left[R_{t+1}, C_{t+1}\right]}{\text{Cov}_t\left[R_{mt+1}, C_{t+1}\right]}(E_t[R_{mt+1}] - R_{ft+1})$$

- return:

$$R_{mt+1} = \frac{P_{mt+1} + D_{mt+1}}{P_{mt}} = D_{mt+1}\frac{\frac{P_{mt+1}}{D_{mt+1}} + 1}{P_{mt}} = C_{t+1}\frac{D_{mt+1}}{C_{t+1}}\frac{\frac{P_{mt+1}}{D_{mt+1}} + 1}{P_{mt}}$$

- If D_{mt}/C_t is constant, we get the CAPM:

$$E_t[R_{t+1}] - R_{ft+1} = \frac{\text{Cov}_t[R_{t+1}, R_{mt+1}]}{\text{Var}_t[R_{mt+1}]}(E_t[R_{mt+1}] - R_{ft+1})$$

Expected Utility, Normal Distribution

- Stein's lemma: if X, Y jointly normally distributed then

$$\mathrm{Cov}[f(X), Y] = \mathrm{E}[f'(X)] \times \mathrm{Cov}[X, Y]$$

- suppose consumption growth and returns are jointly normal
- Then:

$$\mathrm{E}_t[R_{t+1}] - R_{ft+1} = \underbrace{\frac{\mathrm{E}_t[u''(C_{t+1})]\mathrm{Cov}_t[R_{t+1}, C_{t+1}]}{\mathrm{E}_t[u''(C_{t+1})]\mathrm{Cov}_t[R_{mt+1}, C_{t+1}]}}_{\frac{\mathrm{Cov}_t[R_{t+1}, C_{t+1}]}{\mathrm{Cov}_t[R_{mt+1}, C_{t+1}]}} (\mathrm{E}_t[R_{mt+1}] - R_{ft+1})$$