ICAPM

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Assumptions

- Individuals maximize time-additive expected utility:

$$U_t(W_t) = \sum_{j=0}^{\infty} \delta^j u(C_{t+j})$$

- Labor income is tradable
- All retuns iid

$$\approx U_{t+1}(W_t) + \frac{\partial U_{t+1}(W_t)}{\partial W}\Delta W_{t+1} + \frac{1}{2}\frac{\partial U^2_{t+1}(W_t)}{\partial^2 W}(\Delta W_{t+1})^2$$

$$U_t(W_t) = u(C_t) + \delta\, \mathrm{E}_t[U_{t+1}(W_{t+1})]$$

$$\approx U_{t+1}(W_t) + \frac{\partial U_{t+1}(W_t)}{\partial W}\mathrm{E}_t[\Delta W_{t+1}] + \frac{1}{2}\frac{\partial U^2_{t+1}(W_t)}{\partial^2 W}\mathrm{E}_t[(\Delta W_{t+1})^2]$$

$$-W_t + I_t[R_{ft+1} + \sum_a h_a(R_{at+1} - R_{ft+1})]$$

$$\mathrm{Var}_t[\Delta W_{t+1}] + \mathrm{E}_t[\Delta W_{t+1}]^2$$

$$I_t^2 \sum_a \sum_b h_a h_b \mathrm{Cov}_t[R_{at+1}, R_{bt+1}]$$

For short time periods: $I_t \approx W_t$, $\mathrm{E}_t[\Delta W_{t+1}]^2 \approx 0$

First Order Conditions

- FOC with respect to portfolio weight h_{at} (and dividing by W_t):

$$\underbrace{\frac{\partial U_{t+1}(W_t)}{\partial W}}_{U_W} E_t[R_{at+1} - R_{ft+1}] + \underbrace{\frac{\partial U^2_{t+1}(W_t)}{\partial^2 W}}_{U_{WW}} W_t \sum_b h_b \underbrace{\text{Cov}_t[R_{at+1}, R_{bt+1}]}_{\sigma_{ab}} \approx 0$$

- Take the risk-free asset as asset 1:

$$U_W E_t \left[\begin{pmatrix} R_{2t+1} \\ \vdots \\ R_{At+1} \end{pmatrix} - \begin{pmatrix} R_{ft+1} \\ \vdots \\ R_{ft+1} \end{pmatrix} \right] + U_{WW} W_t \begin{pmatrix} \sigma_{22} & \dots & \sigma_{2A} \\ \vdots & & \vdots \\ \sigma_{A2} & \dots & \sigma_{AA} \end{pmatrix} \begin{pmatrix} h_{2t} \\ \vdots \\ h_{At} \end{pmatrix} \approx 0$$

Optimal Portfolio Weights

- Hence

$$\begin{pmatrix} h_{2t} \\ \vdots \\ h_{At} \end{pmatrix} W_t \approx -\frac{U_W}{U_{WW}} \begin{pmatrix} \sigma_{22} & \dots & \sigma_{2A} \\ \vdots & & \vdots \\ \sigma_{A2} & \dots & \sigma_{AA} \end{pmatrix}^{-1} E_t \left[\begin{pmatrix} R_{2t+1} \\ \vdots \\ R_{At+1} \end{pmatrix} - \begin{pmatrix} R_{ft+1} \\ \vdots \\ R_{ft+1} \end{pmatrix} \right]$$

- Hence

$$h_{at} W_t \approx -\frac{U_W}{U_{WW}} \underbrace{(\psi_{a2} \dots \psi_{aA})} E_t \left[\begin{pmatrix} r_{2t+1} \\ \vdots \\ r_{At+1} \end{pmatrix} - \begin{pmatrix} r_{ft+1} \\ \vdots \\ r_{ft+1} \end{pmatrix} \right]$$

ath row of the inverse covariance matrix

Risk Premium

- Define the standardized weight for risky asset a:

$$\tilde{h}_{at} = \frac{h_{at}}{\sum_{j=2}^{A} h_{jt}} \qquad \Rightarrow \qquad \sum_{a=2}^{A} \tilde{h}_{at} = 1$$

- From FOC:

$$\begin{aligned} \mathrm{E}_t[R_{at+1}] - R_{ft+1} &\approx -\frac{U_{WW}}{U_W} W_t \sum_{b=2}^{A} h_{bt}\sigma_{ab} \\ &= -\frac{U_{WW}}{U_W} W_t \left(\sum_{j=2}^{A} h_{jt} \right) \sum_{b=2}^{A} \tilde{h}_{bt}\sigma_{ab} \end{aligned}$$

Putting it all together

- Multiplying by the standardized weights $\tilde{h}_{at}$:

$$\tilde{h}_{at}\mathrm{E}_t[R_{at+1}] - \tilde{h}_{at}R_{ft+1} \approx -\frac{U_{WW}}{U_W}W_t\left(\sum_{j=2}^{A} h_{bt}\right)\sum_{b=2}^{A}\tilde{h}_{at}\tilde{h}_{bt}\sigma_{ab}$$

$$\underset{\sum_a}{\Rightarrow} \quad \underbrace{\mathrm{E}_t[R_{pt+1}]}_{\text{optimal portfolio of risky assets}} - R_{ft+1} \approx -\frac{U_{WW}}{U_W}W_t\left(\sum_{j=2}^{A} h_{jt}\right)\underbrace{\sum_{a=2}^{A}\sum_{b=2}^{A}\tilde{h}_{at}\tilde{h}_{bt}\sigma_{ab}}_{\mathrm{Var}_t[R_{pt+1}]}$$

- Combining with the last equation on the previous slide:

$$\mathrm{E}_t[R_{at+1}] - R_{ft+1} \approx \frac{\sum_{b=2}^{A}\tilde{h}_{bt}\sigma_{ab}}{\sum_{a=2}^{A}\sum_{b=2}^{A}\tilde{h}_{at}\tilde{h}_{bt}\sigma_{ab}}\left(\mathrm{E}_t[R_{pt+1}] - R_{ft+1}\right)$$