Linear Factor Models

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Factors

- factors are random variables $f_{1t}, \ldots, f_{Jt}$
- factors might or might not be traded
- factor space:

$$\text{factor space}(q_t) = \left\{ Y : Y = b_0 + \sum_{j=1}^{J} b_j f_j(q_{t+1}) \text{ for some } b_0, \ldots, b_J \in \mathbb{R} \right\}$$

Orthogonal Factors

- suppose $J = 2$:

$$\text{project } f_1 \text{ on } 1: \quad f_1 = b + z_1 \quad \Longrightarrow \quad \mathrm{E}[z_1] = \mathrm{E}[z_1 1] = 0$$

$$\text{project } f_2 \text{ on } \{1, f_1\}: \quad f_2 = c_1 + c_2 f_1 + z_2 \quad \Longrightarrow \quad \mathrm{E}[z_2] = 0, \quad \mathrm{E}[z_2 z_1] = \mathrm{E}[z_2 f_1] - b\mathrm{E}[z_2] = 0$$

Projecting Payoffs on Factors I

- choose arbitrary payoff Y
- project Y on the factor space:

$$Y_{t+1} = b_0 + \sum_j b_j f_{jt+1} + \overbrace{c_{t+1}}^{\mathrm{E}_t[c_{t+1}]=0,\ \ \mathrm{E}_t[c_{t+1} f_{jt+1}]=0}$$

$$P_{Yt} = E_t[M_{t+1} Y_{t+1}] = b_0 \mathrm{E}_t[M_{t+1}] + \sum_j b_j E_t[M_{t+1} f_{jt+1}] + \mathrm{E}_t[M_{t+1} c_{t+1}]$$

$$R_{Yt+1} = \frac{Y_{t+1}}{P_{Yt}} = \underbrace{\beta_0}_{\frac{b_0}{P_{Yt}}} + \sum_j \underbrace{\beta_j}_{\frac{b_j}{P_{Yt}}} f_{jt+1} + \underbrace{z_{t+1}}_{\frac{c_{t+1}}{P_{Yt}}}$$

Expected Return

- General expected return::

$$\mathrm{E}_t[R_{Yt+1}] = \beta_0 + \sum_j \beta_j \mathrm{E}_t[f_{jt+1}]$$

$$\beta_0 = \frac{1}{\mathrm{E}_t[M_{t+1}]} + \sum_j \beta_j \left(-\frac{\mathrm{E}_t[M_{t+1} f_{jt+1}]}{\mathrm{E}_t[M_{t+1}]} \right) + \left(-\frac{\mathrm{E}_t[M_{t+1} z_{t+1}]}{\mathrm{E}_t[M_{t+1}]} \right)$$

$$\mathrm{E}_t[R_{Yt+1}] = \frac{1}{\mathrm{E}_t[M_{t+1}]} + \sum_j \beta_j \underbrace{\left(-\frac{\mathrm{E}_t[M_{t+1} f_{jt+1}] - \mathrm{E}_t[M_{t+1}]\mathrm{E}_t[f_{jt+1}]}{\mathrm{E}_t[M_{t+1}]} \right)}_{\text{“risk premium of factor } j\text{”}} + \left(-\frac{\mathrm{E}_t[M_{t+1} z_{t+1}]}{\mathrm{E}_t[M_{t+1}]} \right)$$

- If factors have zero price (under M):

$$\mathrm{E}_t[R_{Yt+1}] = \frac{1}{\mathrm{E}_t[M_{t+1}]} + \underbrace{\sum_j \beta_j \mathrm{E}_t[f_{jt+1}] + \left(-\frac{\mathrm{E}_t[M_{t+1} z_{t+1}]}{\mathrm{E}_t[M_{t+1}]} \right)}_{\text{risk premium}}$$

Factors versus Discount Factor

- risk premium from previous slide:

$$\sum_j \beta_j \mathrm{E}_t[f_{jt+1}] + \sum_j \beta_j \left(-\frac{\mathrm{E}_t[M_{t+1} f_{jt+1}]}{\mathrm{E}_t[M_{t+1}]} \right) + \left(-\frac{\mathrm{E}_t[M_{t+1} z_{t+1}]}{\mathrm{E}_t[M_{t+1}]} \right)$$

$$= -\frac{\mathrm{E}_t\left[M_{t+1} \overbrace{\sum_j \left(\beta_j (f_{jt+1} - \mathrm{E}_t[f_{jt+1}]) + z_{t+1} \right)}^{R_{Yt+1} - E_t[R_{Yt+1}]} \right]}{\mathrm{E}_t[M_{t+1}]}$$

Exact Factor Pricing

- suppose M_{t+1} lies in the factor space

$$M_{t+1} = b_{M0} + \sum_j b_{Mj} f_{jt+1}$$

$$\Longrightarrow \underbrace{\mathrm{E}_t[M_{t+1} c_{t+1}]}_{\text{approximation error}} = 0 \quad \Longrightarrow \quad \mathrm{E}_t[R_{Yt+1}] = \frac{1}{\mathrm{E}_t[M_{t+1}]} + \sum_j \beta_j \left(-\frac{\mathrm{Cov}_t[M_{t+1} f_{jt+1}]}{\mathrm{E}_t[M_{t+1}]} \right)$$

Projecting the Discount Factor on the Factor Space

- Project discount factor on factor space:

$$M_{t+1} = a_M + \sum_{j=1}^{J} b_{Mj} f_{jt+1} + z_{Mt+1}$$

- Risk premium:

$$\mathrm{E}_t[R_{t+1}] - \frac{1}{\mathrm{E}_t[M_{t+1}]} = -\frac{\mathrm{Cov}_t[R_{t+1}, M_{t+1}]}{\mathrm{E}_t[M_{t+1}]} = -\sum_j b_{Mj} \frac{\mathrm{Cov}_t[R_{t+1}, f_{jt+1}]}{\mathrm{E}_t[M_{t+1}]} - \frac{\overbrace{\mathrm{Cov}_t[z_{t+1}, z_{Mt+1}]}^{\mathrm{E}_t[M_{t+1} z_{t+1}]}}{\mathrm{E}_t[M_{t+1}]}$$

Pricing Error Bound I

- let z_{at+1} be the return residual of asset a
- suppose $\mathrm{E}_t[z_{at+1} z_{bt+1}] = 0$, $a \neq b$
- this model is also known as the arbitrage pricing theory (APT)
- we have:

$$\sum_{a=1}^{A} \mathrm{E}_t[M_{pt+1} c_{at+1}]^2 \leq \max_a \mathrm{Var}_t[c_{at+1}] \times \mathrm{Var}_t[M_{pt+1}]$$

- step 1 of proof:

$$\overbrace{M_{pt+1}}^{\sum_a H_a(P_{at+1}+D_{at+1})} = \sum_a H_a b_{a0} + \sum_a H_a \sum_j b_{aj} f_{jt+1} + \sum_a H_a c_{at+1}$$

$$\Longrightarrow \quad \text{Var}_t[M_{pt+1}] = \text{Var}_t\left[\sum_a H_a \sum_j b_{aj} f_{jt+1}\right] + \text{Var}_t\left[\sum_a H_a c_{at+1}\right]$$

Pricing Error Bound III

- step 2 of proof:

$$\mathrm{E}_t[M_{pt+1}c_{at+1}] = \mathrm{E}_t\left[\left(\sum_{i=1}^{A} H_i c_{it+1}\right) c_{at+1}\right] = H_a \mathrm{E}_t[c_{at+1}^2]$$

$$\Longrightarrow \overbrace{\mathrm{E}_t[M_{pt+1}c_{at+1}]^2}^{H_a^2 \mathrm{E}_t[c_{at+1}^2]^2} \leq H_a^2 \mathrm{E}_t[c_{at+1}^2] \times \max_i \mathrm{E}_t[c_{it+1}^2]$$

$$\Longrightarrow \sum_a E_t[M_{pt+1}c_{at+1}]^2 \leq \max_a \underbrace{\mathrm{E}_t[c_{at+1}^2]}_{\mathrm{Var}_t[c_{at+1}]} \times \underbrace{\sum_a H_a^2 \mathrm{E}_t[c_{at+1}^2]}_{\mathrm{E}_t[(\sum_a H_a c_{at+1})^2] = \mathrm{Var}_t[\sum_a H_a c_{at+1}]}$$

$$\leq \max_a \mathrm{Var}_t[c_{at+1}]\mathrm{Var}_t[M_{pt+1}]$$