Complete Market

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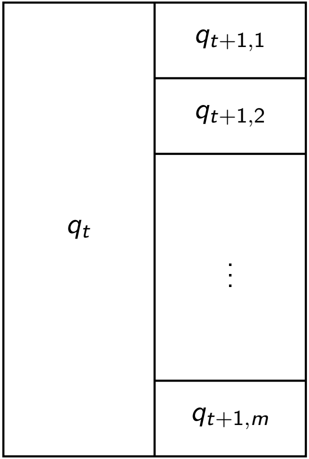
Local Market

- local market:



- global market: all local markets

Dropping Assets that do not Exist in the Local Market

- suppose the following assets exist in q_t: $\{a_1, a_2, \ldots, a_{A(q_t)}\}$
- payoff matrix:

$$\mathbf{P}^*_{t+1}(q_t) = \underbrace{\begin{pmatrix} P_{a_1}(q_{1t+1}) + D_{a_1}(q_{1t+1}) & \cdots & P_{a_A}(q_{1t+1}) + D_{a_A}(q_{1t+1}) \\ \vdots & \ddots & \vdots \\ P_{a_1}(q_{mt+1}) + D_{a_1}(q_{mt+1}) & \cdots & P_{a_A}(q_{mt+1}) + D_{a_A}(q_{mt+1}) \end{pmatrix}}_{m \times A}$$

- portfolio payoff:

$$\mathbf{P}^*_{t+1}(q_t) \cdot \mathbf{H}(q_t) = \begin{pmatrix} \Big(\mathbf{P}(q_{1t+1}) + \mathbf{D}(q_{1t+1})\Big) \cdot \mathbf{H}(q_t) \\ \vdots \\ \Big(\mathbf{P}(q_{mt+1}) + \mathbf{D}(q_{mt+1})\Big) \cdot \mathbf{H}(q_t) \end{pmatrix}$$

Local Law of One Price

- portfolios with the same payouts have the same prices:

$$\underbrace{\mathbf{P}^*_{t+1}\mathbf{H}_1}_{\text{payoff 1}} = \underbrace{\mathbf{P}^*_{t+1}\mathbf{H}_2}_{\text{payoff 2}} \quad \Longrightarrow \quad \underbrace{\mathbf{P}_{H_1 t}}_{\text{price 1}} = \underbrace{\mathbf{P}_{H_2 t}}_{\text{price 2}} \quad \text{for all } H_1, H_2.$$

Removing Redundant Assets

- example: suppose in the local market q_t there are $m = 3$ future states and $A = 3$ assets with payoffs:

	Assets		
State	a_1	a_2	a_3
1	1	2	5
2	1	2	5
3	1	3	7

Payoff Space

- payoff space in market q_t is the column space of the payoff matrix $\mathbf{P}^*_{t+1}(q_t)$:

  $$\text{payoff space } (q_t) = \{\mathbf{x} : \mathbf{x} = \mathbf{P}^*_{t+1}(q_t)\mathbf{H} \text{ for some portfolio } \mathbf{H}\}.$$

Locally Complete Markets

- market q_t is complete if the payoff space in q_t has dimension m
- hence in a complete market we can generate any arbitrary $t+1$ payoff $\mathbf{y}$ through some linear combination of traded asset:

$$\underbrace{\mathbf{P}^*_{t+1}}_{m \times m \text{ since we dropped redundant assets}} \mathbf{H} = \mathbf{x} \qquad \Longrightarrow \qquad \mathbf{H} = (\mathbf{P}^*_{t+1})^{-1}\mathbf{x}.$$

State Assets

- state asset (or Arrow-Debreu security) q_τ:

$$D_t = \begin{cases} 1 & \text{if } t = \tau \text{ and } s \in q_\tau \\ 0 & \text{otherwise} \end{cases}$$

- prices of local state assets (if they exist):

$$\mathbf{P}_{q_{t+1}}(q_t) = \begin{pmatrix} P_{q_{1t+1}}(q_t) \\ \vdots \\ P_{q_{mt+1}}(q_t) \end{pmatrix}$$

Replicating Payoffs with State Assets

- if all state assets are traded we can replicate any arbitrary payoff $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \#\text{ of state asset 1} \\ \vdots \\ \#\text{ of state asset m} \end{pmatrix}$$

- hence:

$$\text{market complete} \quad \Longleftrightarrow \quad \text{all state assets exist}$$

Prices of State Assets in Locally Complete Markets I

- if market q_t is complete:

$$\underbrace{\mathbf{P}_{t+1}^{*}\,\mathbf{H}}_{\text{portfolio replicating state asset 1}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{P}_{t+1}^{*} \underbrace{\left(\mathbf{H}_1 \cdots \mathbf{H}_m\right)}_{\text{matrix containing all replicating portfolios as column vectors}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Prices of State Assets in Locally Complete Markets II

- multiplying with $\mathbf{P}_t$:

$$\left(\mathbf{H}_1 \cdots \mathbf{H}_m\right) = \left(\mathbf{P}^*_{t+1}\right)^{-1} \quad \Longrightarrow \quad \overbrace{\left(\mathbf{H}_1 \cdots \mathbf{H}_m\right)^T \cdot \mathbf{P}_t}^{\text{prices of state assets}} = \left(\left(\mathbf{P}^*_{t+1}\right)^{-1}\right)^T \cdot \overbrace{\mathbf{P}_t}^{\text{asset prices}}$$

$$\Longrightarrow \quad \mathbf{P}_{q_{t+1}}(q_t) = \left(\left(\mathbf{P}^*_{t+1}\right)^{-1}\right)^T \cdot \mathbf{P}_t$$

Using Prices of State Asset to Find Prices of General Assets

- Solving for the asset price vector:

$$\underbrace{\mathbf{P}_t}_{\text{prices}} = \underbrace{(\mathbf{P}^*_{t+1})}_{\text{payoffs}}{}^T \underbrace{\mathbf{P}_{q_{t+1}}(q_t)}_{\text{prices of state assets}}$$

- hence we have for each asset a:

$$P_a(q_t) = \sum_{q_{t+1} \subseteq q_t} P_{q_{t+1}}(q_t) \times \Big(P_a(q_{t+1}) + D_a(q_{t+1})\Big)$$

Globally Complete Markets I

- market globally complete: all state assets are traded at time $t = 0$
- consider a sequence of sub-partition elements

$$q_t \supseteq q_{t+1} \supseteq q_{t+2}$$

- suppose each local market in this sequence is complete
- "rolling over" the two short-term state assets:
  1. At q_t we purchase $P_{q_{t+2}}(q_{t+1})$ units of q_{t+1} for the price of $P_{q_{t+1}}(q_t)$ per unit.
  2. At q_{t+1} we use the proceeds from (1) to purchase one unit of q_{t+2}.

Globally Complete Markets II

- can construct a long-term state asset for any sequence of locally complete markets:

$$q_t \supseteq q_{t+1} \supseteq \cdots \supseteq q_{t+\tau}.$$

- price of the long-term state asset:

$$P_{q_{t+\tau}}(q_t) = P_{q_{t+1}}(q_t) \times P_{q_{t+2}}(q_{t+1}) \times \cdots \times P_{q_{t+\tau}}(q_{t+\tau-1})$$

- this price is unique
- we have:

global market complete $\iff$ all local markets complete