Arbitrage, Risk-Neutral Probabilities & Discount Factor

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Local Arbitrage Opportunity

- local arbitrage opportunity in market q_t:

$$0 \neq \left\{ -P_H(q_t), \begin{pmatrix} P_H(q_{1t+1}) + D_H(q_{1t+1}) \\ \vdots \\ P_H(q_{mt+1}) + D_H(q_{mt+1}) \end{pmatrix} \right\} \geq 0, \qquad q_{it+1} \subseteq q_t$$

- or in random variable notation:

$$0 \neq \{-P_{Ht}, P_{Ht+1} + D_{Ht+1}\} \geq 0$$

Longterm Arbitrage

- finite longterm arbitrage opportunity:

$$0 \neq \{-P_H(q_t), D_{Ht+1}, \dots, D_{Ht+\tau} + P_{Ht+\tau}\} \geq 0$$

- infinite longterm arbitrage opportunity:

$$0 \neq \{-P_H(q_t), D_{Ht+1}, D_{Ht+2}, \dots\} \geq 0$$

Local versus Global Arbitrage

- by definition of long-term arbitrage:

$$\text{local arbitrage} \quad \Longrightarrow \quad \text{longterm arbitrage}$$

- backward induction:

$$\text{no local arbitrage} \quad \Longrightarrow \quad \text{no finite longterm arbitrage}$$

- but:

$$\text{no local arbitrage} \quad \not\Rightarrow \quad \text{no infinite longterm arbitrage}$$

- no perpetual borrowing condition:

$$P_{Ht} < 0 \text{ or } \underbrace{0 \neq D_{Ht+\tau_1} \geq 0}_{> 0 \text{ in at least one } q_{t+\tau_1}} \quad \Longrightarrow \quad P_{Ht+\tau_2} \geq 0 \text{ for some } \tau_2 \geq \tau_1$$

Arbitrage & Law of One Price

$$\text{no arbitrage} \quad \Longrightarrow \quad \text{law of one price holds}$$

Arbitrage & State Prices

- we have:

  there exist a state price vector $> 0 \quad \Longrightarrow \quad$ no arbitrage

- proof:

$$P_H(q_t) = \sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t) \left[P_H(q_{t+1}) + D_H(q_{t+1}) \right]$$

Arbitrage & State Prices: Complete Market

$$\text{no arbitrage} \quad \Longleftrightarrow \quad \text{state prices} \; > 0$$

Risk Neutral Probabilities I

- law of one price:

$$P_H(q_t)$$
$$= \sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t) \times \Big(P_H(q_{t+1}) + D_H(q_{t+1}) \Big)$$
$$= \left(\sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t) \right) \sum_{q_{t+1} \subseteq q_t} \frac{\pi_{q_{t+1}}(q_t)}{\sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t)} (P_H(q_{t+1}) + D_H(q_{t+1}))$$

- for $x \subseteq \{q_{1t+1}, \ldots, q_{mt+1}\}$ define

$$A_x(q_t) = \sum_{q_{t+1} \in x} \frac{\pi_{q_{t+1}}(q_t)}{\sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t)}.$$

Risk Neutral Probabilities II

- then:
  1. $A_x(q_t) \geq 0 \quad$ (no abitrage)
  2. $x_1 \cap x_2 = \emptyset \Rightarrow A_{x_1 \cup x_2}(q_t) = A_{x_1}(q_t) + A_{x_2}(q_t)$
  3. $\sum_{q_{t+1} \subseteq q_t} A_{q_{t+1}}(q_t) = 1$
- define expectation with respect to A as

$$\mathrm{E}^A[Y_{t+1}|q_t] = \sum_{q_{t+1} \subseteq q_t} A_{q_{t+1}}(q_t) \times Y(q_{t+1}).$$

- hence:

$$P_H(q_t) = \left(\sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t) \right) \times \mathrm{E}^A \left[\left(P_H(q_{t+1}) + D_H(q_{t+1}) \right) \middle| q_t \right]$$

- if a risk-free asset exists:

$$P_H(q_t) = \frac{1}{R_{ft+1}} \mathrm{E}^A \left[\left(P_H(q_{t+1}) + D_H(q_{t+1}) \right) \middle| q_t \right]$$

The Stochastic Discount Factor I

For each local market, define:

q_t	q_{1t+1} :	$M_{q_{1t+1}} = \frac{\pi_{q_{1t+1}}(q_t)}{\text{prob}_{q_{1t+1}}(q_t)}$
	q_{2t+1} :	$M_{q_{2t+1}} = \frac{\pi_{q_{2t+1}}(q_t)}{\text{prob}_{q_{2t+1}}(q_t)}$
		$\vdots$
	q_{mt+1} :	$M_{q_{mt+1}} = \frac{\pi_{q_{mt+1}}(q_t)}{\text{prob}_{q_{mt+1}}(q_t)}$

The Stochastic Discount Factor II

- we have for any portfolio:

$$P_H(q_t) = \sum_{q_{t+1} \subseteq q_t} \pi_{q_{t+1}}(q_t) \times \Big(P_H(q_{t+1}) + D_H(q_{t+1})\Big)$$

$$= \sum_{q_{t+1} \subseteq q_t} \text{prob}_{q_{t+1}}(q_t) \times \frac{\pi_{q_{t+1}}(q_t)}{\text{prob}_{q_{t+1}}(q_t)} \times \Big(P_H(q_{t+1}) + D_H(q_{t+1})\Big)$$

- then:

$$P_H(q_t) = \mathsf{E}\big[M_{t+1}\big(P_H(q_{t+1}) + D_H(q_{t+1})\big)\big|q_t\big]$$

- any portfolio:

$$P_{Ht} \qquad \mathrm{E}_{t+1}[M_{t+2}(D_{Ht+2} + P_{Ht+2})]$$

$$= \mathrm{E}_t[M_{t+1}(D_{Ht+1} + \overbrace{P_{Ht+1}})]$$

$$= \mathrm{E}_t[M_{t+1}D_{Ht+1}] + \underbrace{\mathrm{E}_t\left[M_{t+1}\left(\mathrm{E}_{t+1}[M_{t+2}D_{Ht+2}] + \mathrm{E}_{t+1}[M_{t+2}P_{Ht+2}]\right)\right]}$$

$$\underbrace{\mathrm{E}_t\left[\mathrm{E}_{t+1}[M_{t+1}M_{t+2}D_{Ht+2}] + \mathrm{E}_{t+1}[M_{t+1}M_{t+2}V_{Ht+2}]\right]}$$

$$\vdots \qquad \mathrm{E}_t[M_{t+1}M_{t+2}D_{Ht+2}] + \mathrm{E}_t[M_{t+1}M_{t+2}D_{Ht+2}]$$

$$= \sum_{j=1}^{\tau} \mathrm{E}_t\left[(M_{t+1}\cdots M_{t+j})D_{Ht+j}\right] + \mathrm{E}_t\left[(M_{t+1}\cdots M_{t+\tau})P_{Ht+\tau}\right]$$

Long-Term Stochastic Discount Factor II

- define long-term discount factor:

$$M_t^{t+j} = M_{t+1} \cdots M_{t+j}$$

- then:

$$P_{Ht} = \sum_{j=1}^{\tau} \mathrm{E}_t[M_t^{t+j} D_{Ht+j}] + \mathrm{E}_t[M_t^{t+\tau} P_{Ht+\tau}]$$

Discount Factor and Returns

- dividing the short-term pricing equation by the current price:

$$1 = \mathrm{E}_t[M_{t+1} R_{t+1}]$$

- Suppose we purchase a self-financing portfolio H at t and sell it at $t + \tau$:

$$P_{Ht} = \mathrm{E}_t[M_t^{t+\tau} P_{Ht} R_{Ht}^{t+\tau}]$$

- hence:

$$1 = \mathrm{E}_t[M_t^{t+\tau} R_t^{t+\tau}]$$